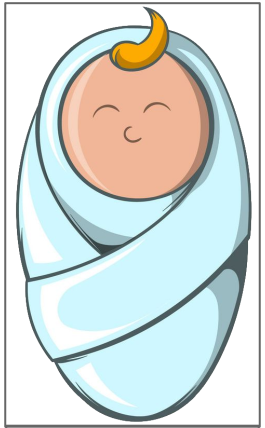
Numerical Methods in LQG

aka: can we compute spin foam amplitudes?

Numerical Methods in LQG

yes! let's see how.

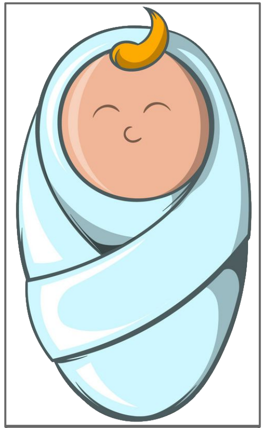
Disclaimer



faithful representation of
numerical methods in LQG

Disclaimer

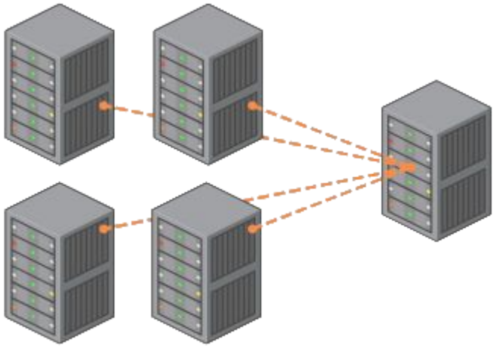
- Very recent
- Requires a lot of time
- Solve "stinky" problems
- Lots of questions and very few answers



faithful representation of numerical methods in LQG

What is this lecture about?

- Who, What, Where? A little bit of how.
- 3D gravity. The Ponzano-Regge model.
- 4D gravity. The EPRL model.
- EPRL vertex amplitude computed with HPC methods



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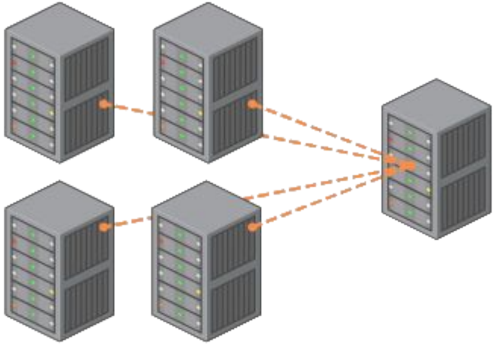
> On which topics(s) would you like lectures for the next LQG school?
>
> Perhaps a couple of lecture where show explicit calculations for transition amplitudes and such. I understand they are not exactly easy to do.



What is this lecture about?

I am sorry for my poor pictures

There will be a lot of happy little accidents

Who, What, Where?

`sl2cfoam and sl2cfoam-next`

3 years - QG group - Marseille

- divide and conquer!
- HPC ready, fast and efficient
- Truncation needed, error estimation, computationally expensive large triangulations
- single vertex, 3 vertices, bubble divergences

P.D., Francesco Gozzini, Giorgio Sarno
+ (Western U) Francesca Vidotto, Carlo Rovelli, Pietropaolo Frisoni, …

Original paper 1807.03066, advanced paper will appear soon
applications: 1803.00835, 1903.12624, 2004.12911 and more to come

Who, What, Where?

Markov Chain Monte-Carlo Computation

1 year - Florida Atlantic University + Fudan

Integration over a Lefschetz Thimble to avoid oscillations

Huge spins are accessible

Not suitable for small spins

Muxin Han, Zichang Huang, Hongguang Liu, Dongxue Qu, Yidun Wan

Main paper 2012.11515

Who, What, Where?

Effective spin foam models

1 year - Bard College and Perimeter Institute

2 hours lecture yesterday

Seth Asante, Bianca Dittrich, Hal Haggard

Euclidean 2004.07013 and Lorentzian 2104.00485

Plus the hands-on experiment NEXT WEEK!

Renormalization and fixed points

6 years - U. Hamburg + U. Jena

Fixed points under coarse graining.
Truncation made of hypercubes (slightly generalized)

Benjamin Bahr and Sebastian Steinhaus

Original paper 1508.07961, numerics 1605.07649, review 2007.01315

3D Gravity: The Ponzano-Regge model (Again)

Semiclassical limit of Racah coefficients - G.Ponzano and T.Regge

Transition amplitudes for Euclidean 3D LQG (discussed in length by Maïté, Edward Wilson-Ewing, and Valentin Bonzom)

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boundary state

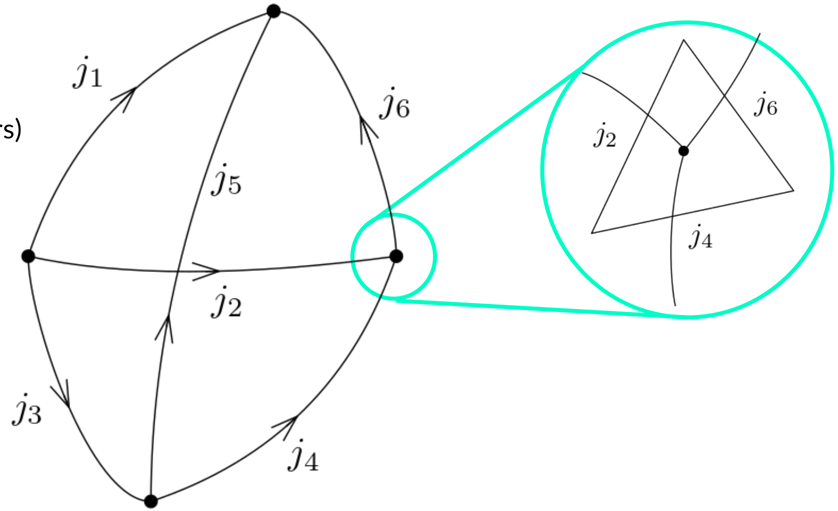
Restrict to 3 valent spinnetworks (oriented graph, irreps, intertwiners)

Spins as lengths eigenvalues

Trivial intertwiners in 3D (1d space) $i^j_m = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$

Nodes = Triangles (closure)

(Quantum) triangulations of a 2D surface (possibly curved)



3D Gravity: The Ponzano-Regge model - The amplitude

Semiclassical limit of Racah coefficients - G.Ponzano and T.Regge

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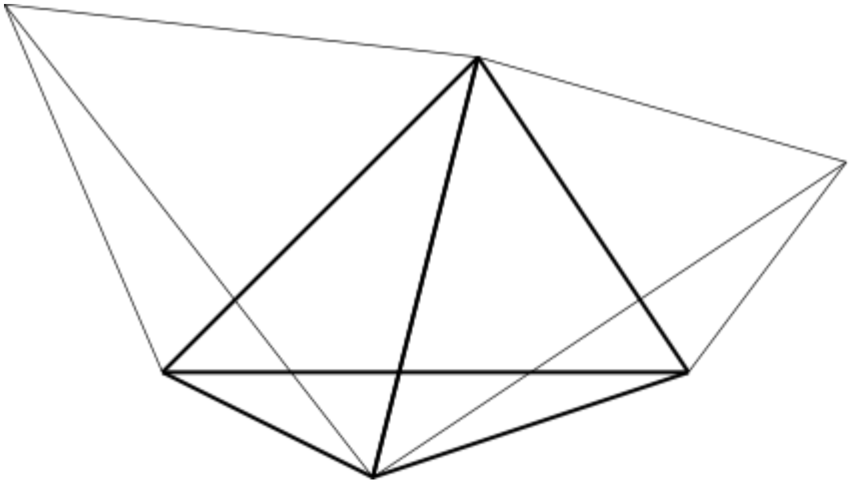
Choose a 2 complex dual to a 3D triangulation (vertices, edges, faces)



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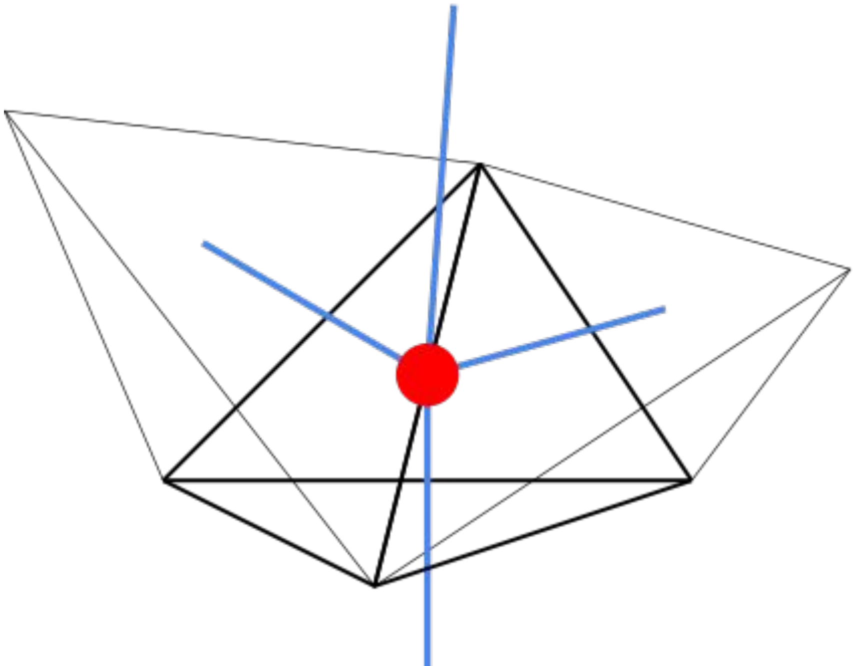

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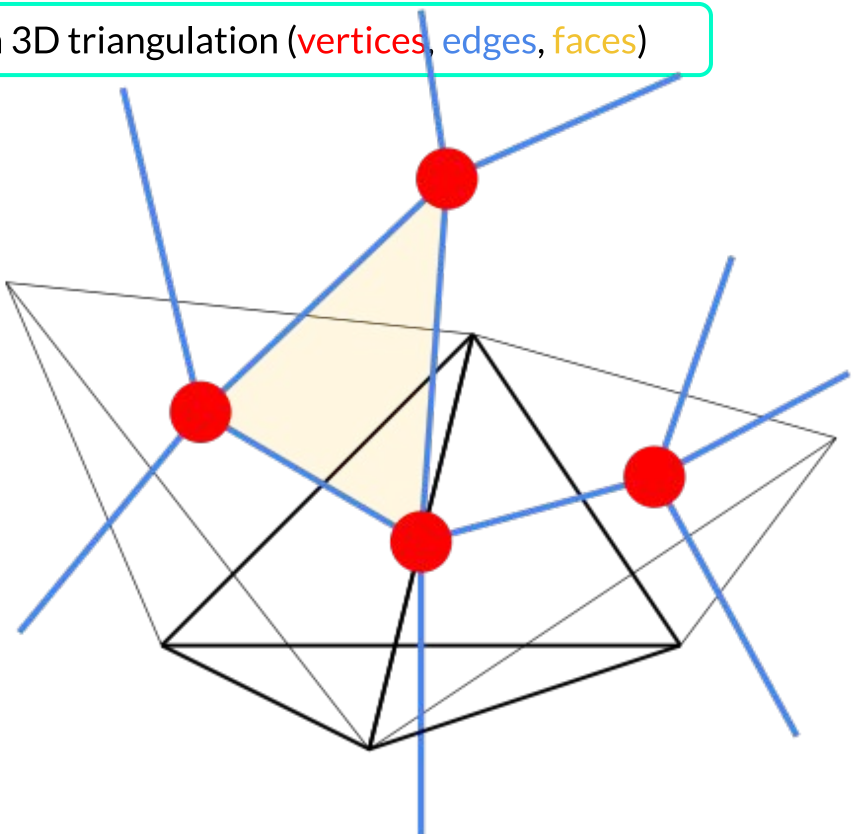
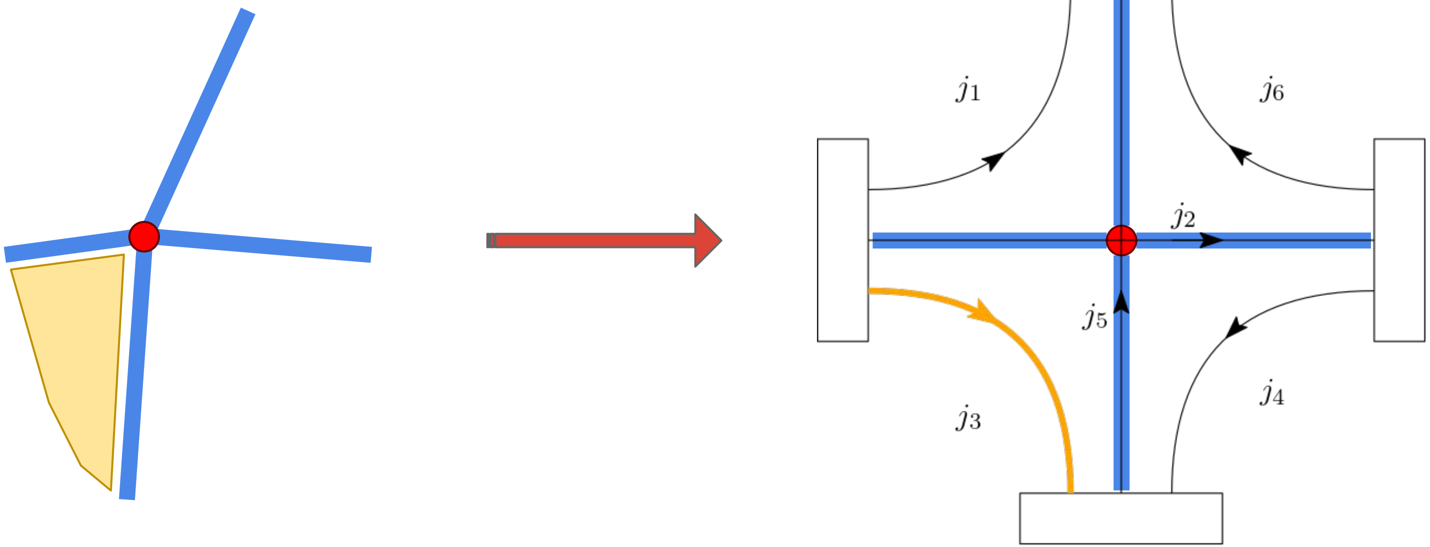


3D Gravity: The Ponzano-Regge model (wiring)

Let me change graphical notation



SU(2) IN ONE SLIDE

Angular Momentum Algebra

$$[J_a, J_b] = i\epsilon_{abc} J_c$$

$$J_3 \left|j, m\right\rangle = m \left|j, m\right\rangle$$

$$J^2 \left|j, m\right\rangle = j(j+1) \left|j, m\right\rangle$$

Group matrix elements in the irrep j

$$\left\langle j, m\right| g \left|j, n\right\rangle \equiv D^j_{mn}(g)$$

Numerics need a parametrization (Euler) - fundamental

$$g = e^{i\phi\frac{\sigma_3}{2}} e^{i\theta\frac{\sigma_3}{2}} e^{i\psi\frac{\sigma_3}{2}}$$

Matrix elements explicit formula

$$D^j_{mn}(g) = e^{-i\phi\frac{m}{2}} e^{i\psi\frac{n}{2}} \sqrt{(j+m)!(j-m)!(j+n)!(j-n)!}$$
$$\sum_s (-1)^{m-n+s} \frac{\left(\cos\frac{\theta}{2}\right)^{2j+n-m-2s} \left(\sin\frac{\theta}{2}\right)^{m-n+2s}}{(j+n-s)!s!(m-n+s)!(j-m-s)!}$$

Integration of functions on the group using the Haar measure

$$\int f(g)\, dg = \frac{1}{16\pi^2} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \int_0^{4\pi} d\psi\, f(g)$$

[A Primer of Group Theory for Loop Quantum Gravity and Spin-foams - Pierre Martin-Dussaud]

[Quantum Theory of Angular Momentum - Varshalovich - aka the sacred text for SU(2) calculations]

3D Gravity: The Ponzano-Regge model - The amplitude

Recipe to write any amplitude:

For each vertex

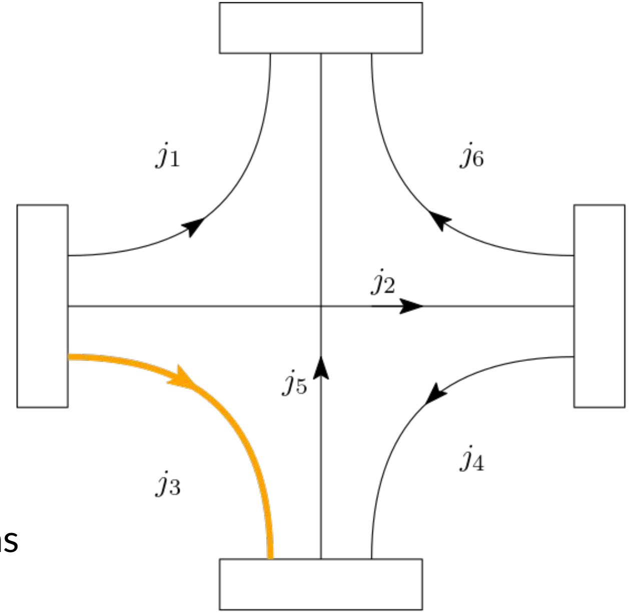
For each edge one group element g_e

For each face an irrep j_f and a matrix element $D^{j_f}_{m_s m_t}(g_s^{-1} g_t)$

Integrate over all the group elements $\int dg_e$

Sum over the internal faces spins with a weight $(2j_f + 1)$

Tell them about the interpretation in terms of reference systems



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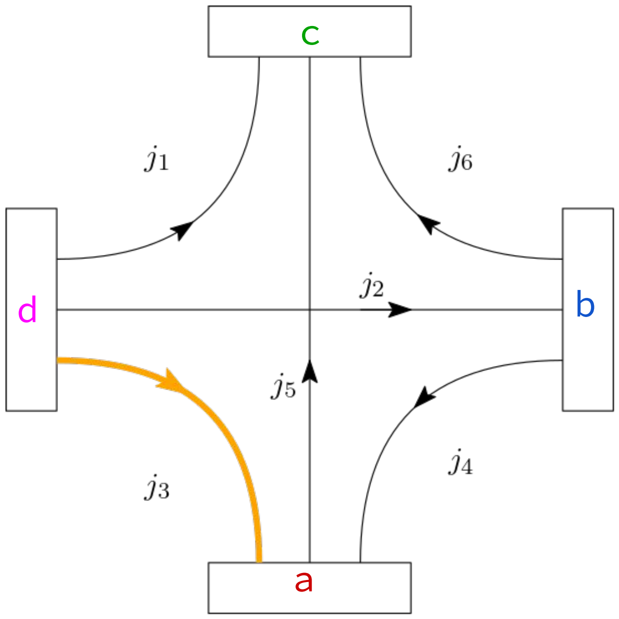

$$D^{j_1}(g_d^{-1} g_c) D^{j_2}(g_d^{-1} g_b) D^{j_3}(g_d^{-1} g_a)$$
$$D^{j_4}(g_a^{-1} g_b) D^{j_5}(g_a^{-1} g_c) D^{j_6}(g_b^{-1} g_c)$$

3D Gravity: The Ponzano-Regge model - The amplitude

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Integrate over all the group elements $\int dg_e$

$$\int dg_a\, dg_b\, dg_c\, dg_d$$
$$D^{j_1}(g_d^{-1} g_c) D^{j_2}(g_d^{-1} g_b) D^{j_3}(g_d^{-1} g_a)$$
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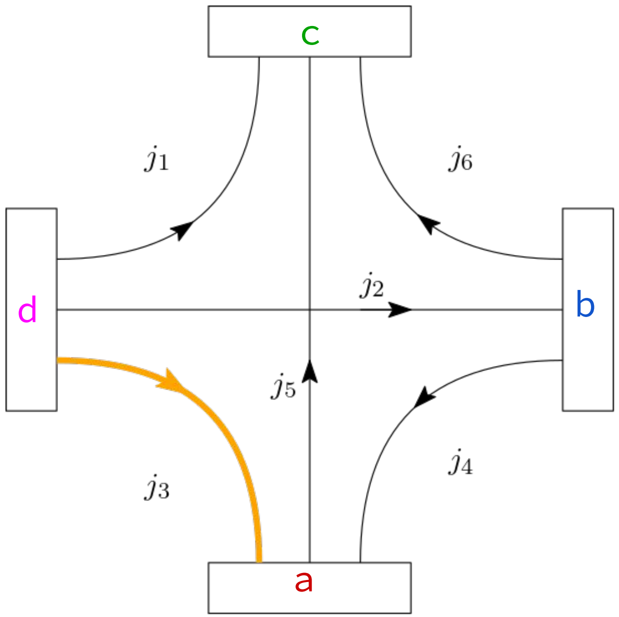

3D Gravity: The Ponzano-Regge model - The amplitude

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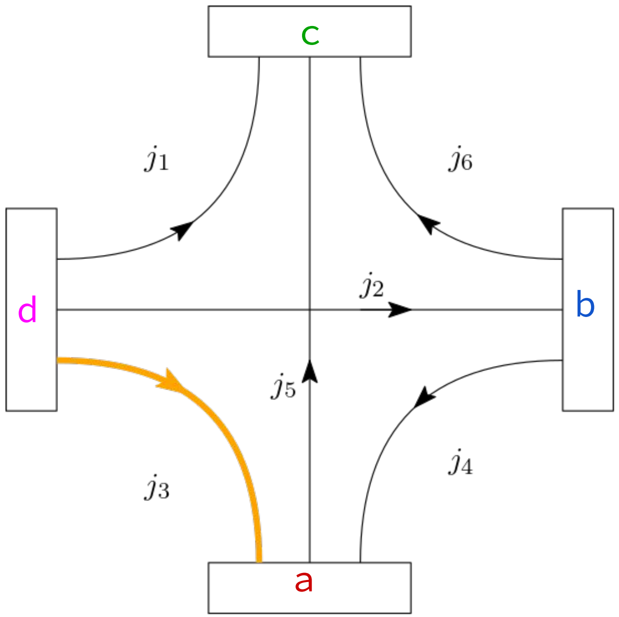
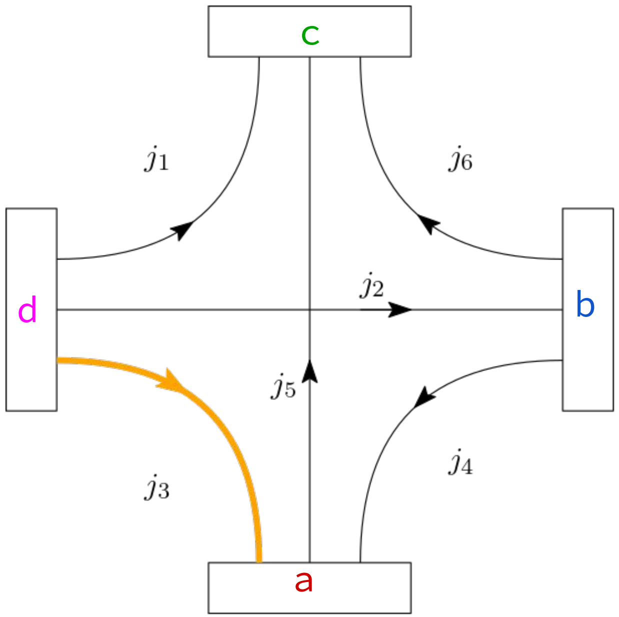


Split the irreps, write the formula for the *D*, 6 sums of elementary integrals...

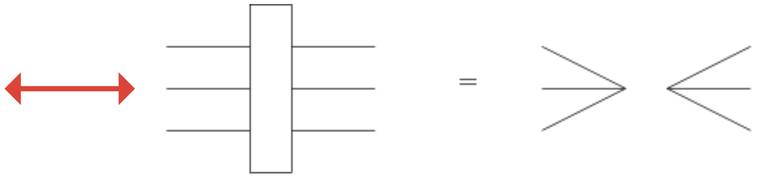
3D Gravity: The Ponzano-Regge model - The amplitude

$$\int dg_a\, dg_b\, dg_c\, dg_d \; D^{j_1}(g_d^{-1} g_c) D^{j_2}(g_d^{-1} g_b) D^{j_3}(g_d^{-1} g_a) D^{j_4}(g_a^{-1} g_b) D^{j_5}(g_a^{-1} g_c) D^{j_6}(g_b^{-1} g_c)$$



Or do the integrals analytically

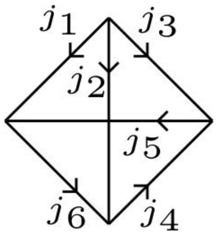
$$\int dg D^{j_1}_{m_1 n_1}(g) D^{j_2}_{m_2 n_2}(g) D^{j_3}_{m_3 n_3}(g) = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{pmatrix}$$



3D Gravity: The Ponzano-Regge model - The amplitude

Each vertex gives you a Wigner $\{6j\}$ symbol.

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}$$



Many explicit expressions: e.g. contraction of CG coefficients, in term of a sum of factorials, Hypergeometric function, etc.

Quantum Theory of Angular Momentum - Varshalovich - aka the sacred text for SU(2) calculations

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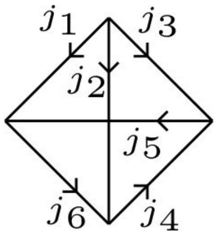

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Numerically:

1. Code your own
2. Do not reinvent the wheel
   - In Mathematica is builtin `SixJSymbol[{j1, j2, j3}, {j4, j5, j6}]`
   - C, python and Matlab use `WIGXJPF`

Fast and Accurate Evaluation of Wigner 3j, 6j, and 9j Symbols Using Prime Factorization and Multiword Integer Arithmetic - H. T. Johansson and C. Forssén

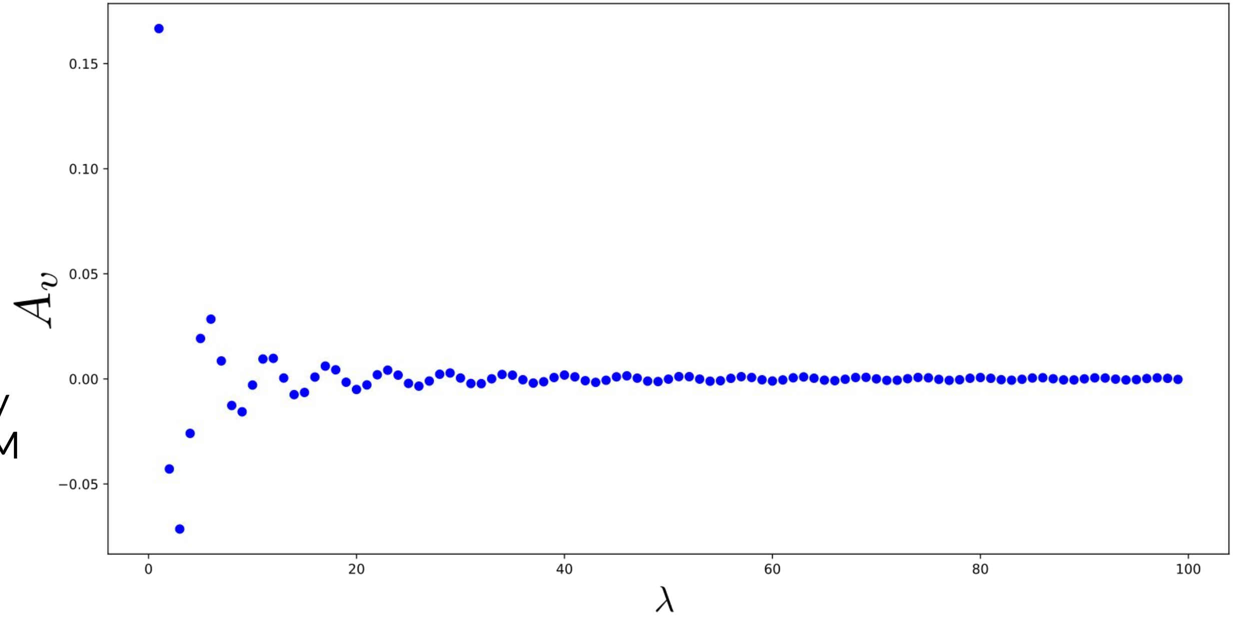
Example files in the school website

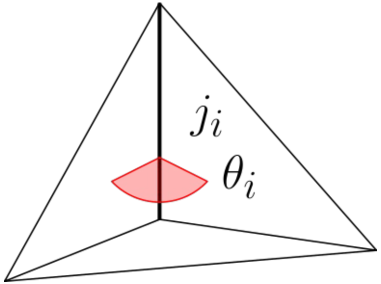
3D Gravity: The Ponzano-Regge model - The asymptotic

Semiclassical limit of Racah coefficients - G.Ponzano and T.Regge

Vertex amplitude for large spins is related to the Regge action

$$A_v = \begin{Bmatrix} \lambda j_1 & \lambda j_2 & \lambda j_3 \\ \lambda j_4 & \lambda j_5 & \lambda j_6 \end{Bmatrix} \approx \frac{1}{\lambda^{\frac{3}{2}}\sqrt{48\pi V}} \left(e^{i\lambda(\sum_i j_i\theta_i)+i\frac{\pi}{4}} + e^{-i\lambda(\sum_i j_i\theta_i)-i\frac{\pi}{4}} \right)$$





See Hal Haggard and
Seth Asante lecture for more details

Analogy with QM path integral

Example: $j_i = 1$

3D Gravity: The Ponzano-Regge model - The asymptotic

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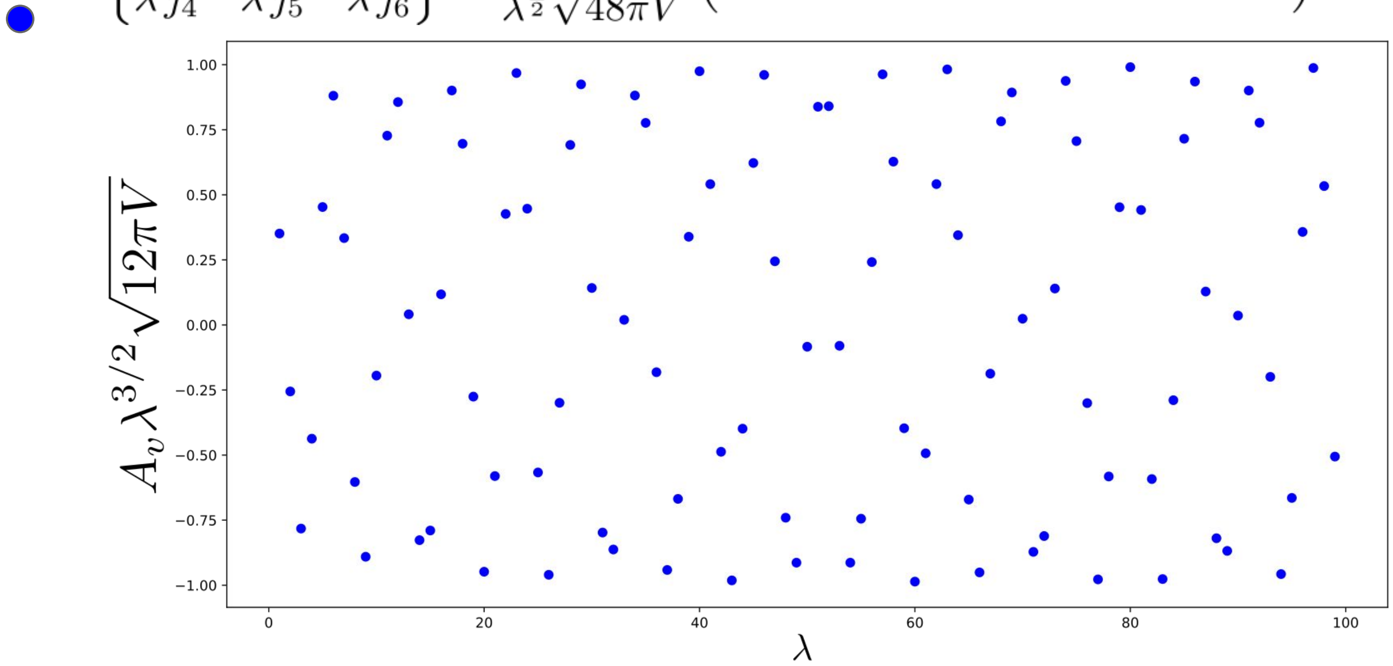

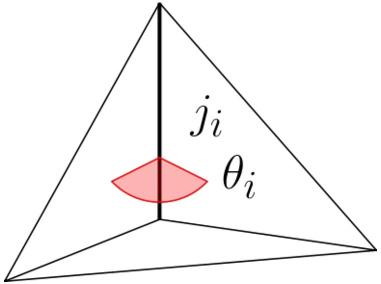

Example: $j_i = 1$

$$\sum_i j_i\theta_i \approx 11.46$$

3D Gravity: The Ponzano-Regge model - The asymptotic

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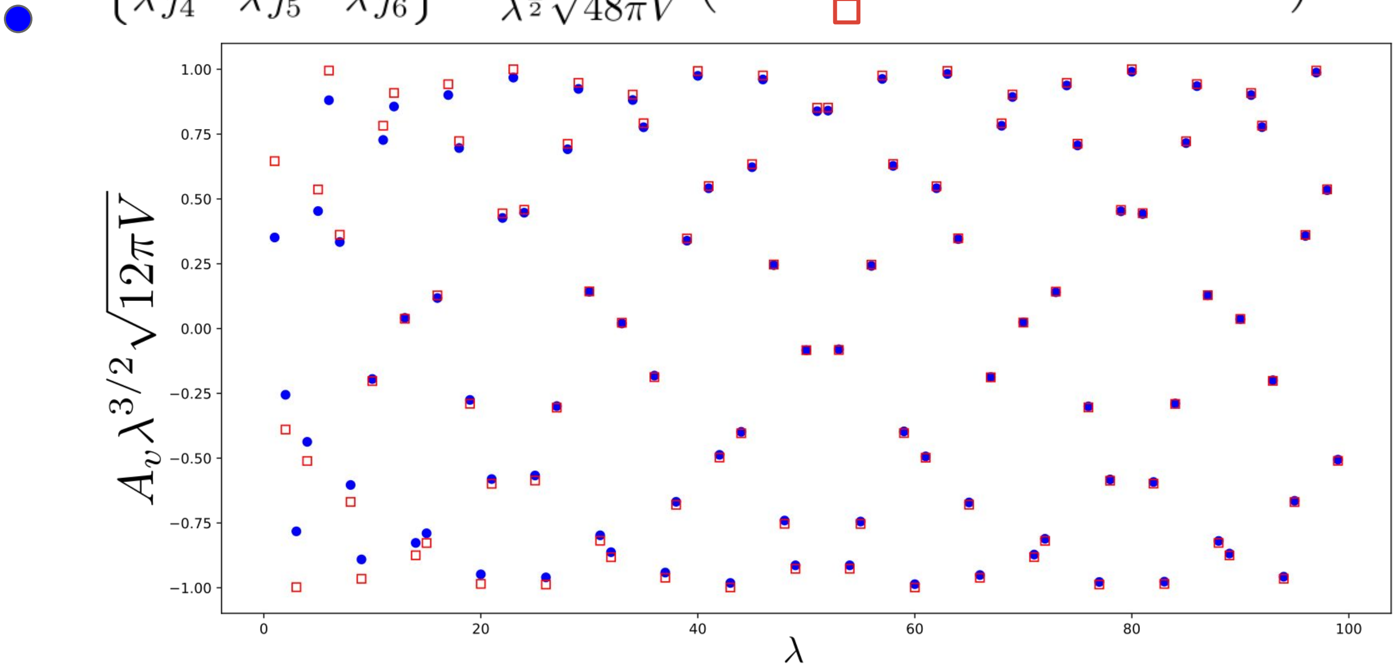

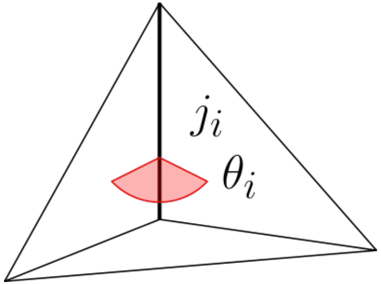

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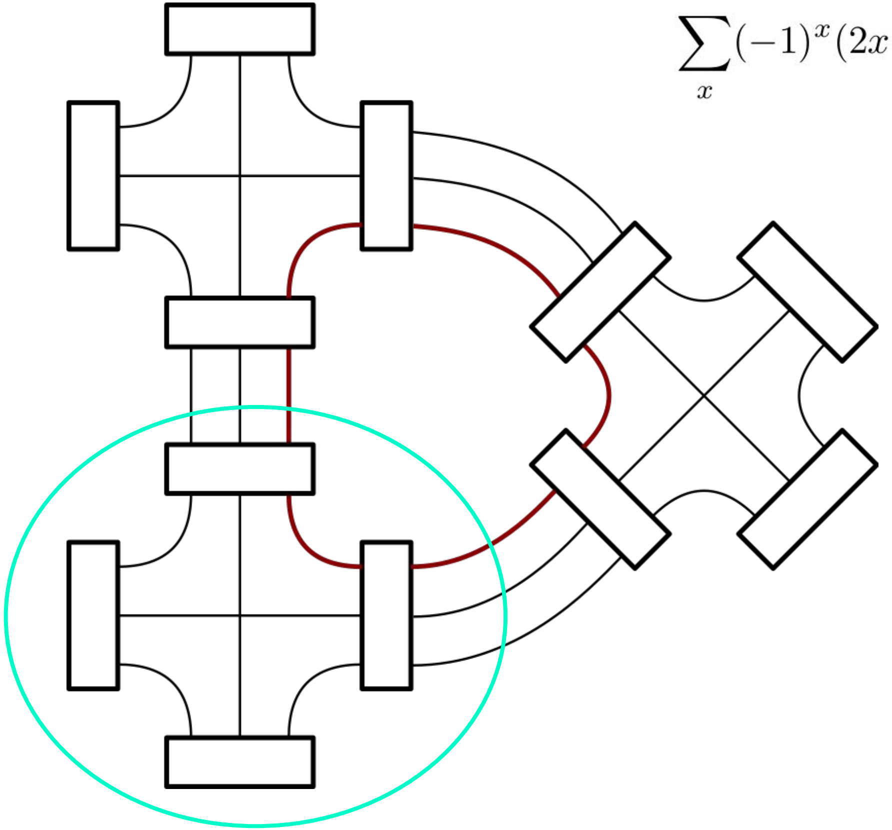
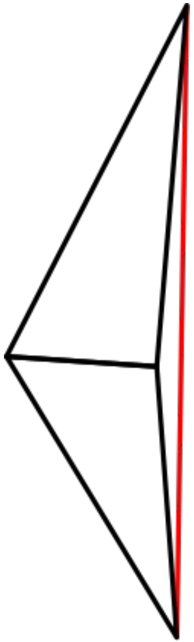
3D Gravity: The Ponzano-Regge model - 3 vertices

$$\sum_x (-1)^x (2x+1) \begin{Bmatrix} j_5 & j_8 & x \\ j_9 & j_6 & j_1 \end{Bmatrix} \begin{Bmatrix} j_9 & j_6 & x \\ j_4 & j_7 & j_2 \end{Bmatrix} \begin{Bmatrix} j_4 & j_7 & x \\ j_8 & j_5 & j_3 \end{Bmatrix}$$





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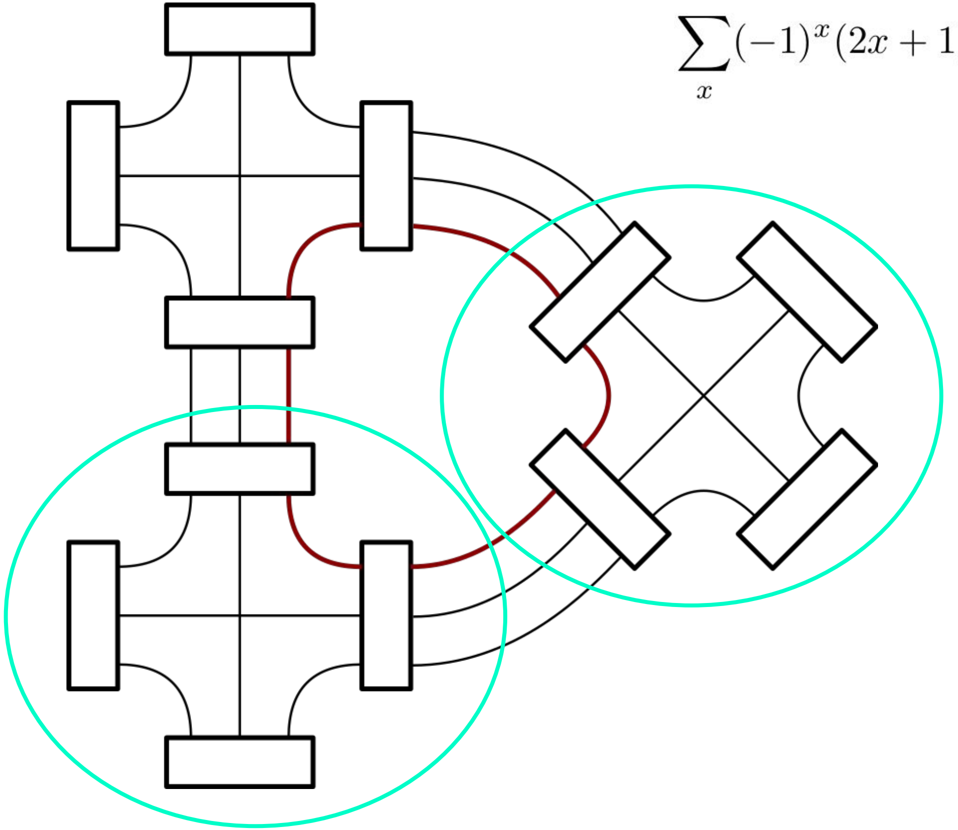

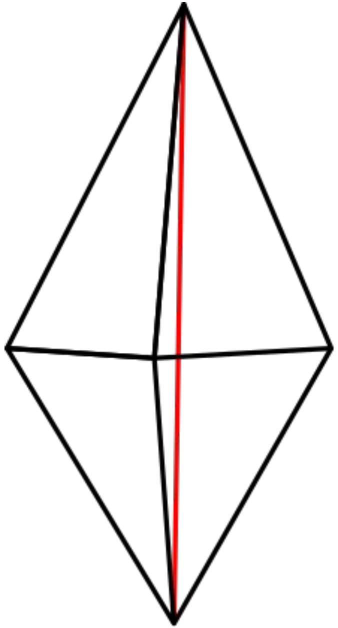

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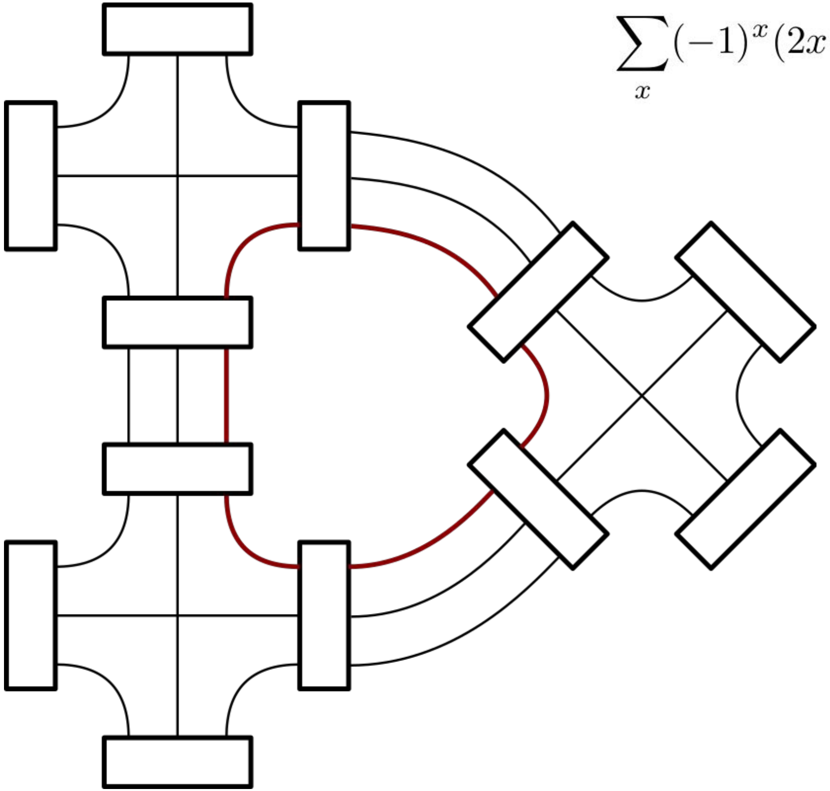

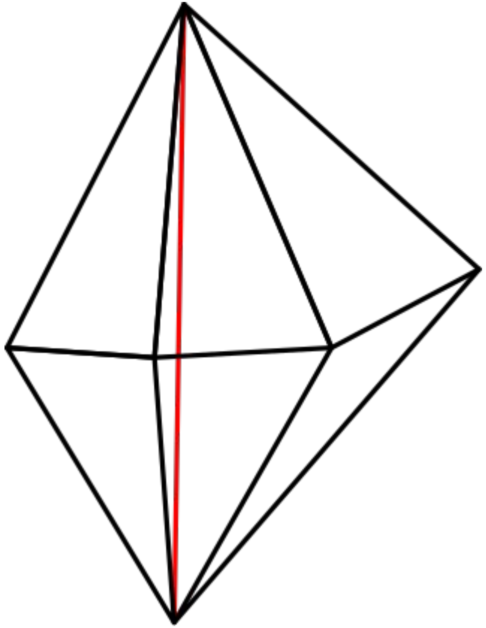

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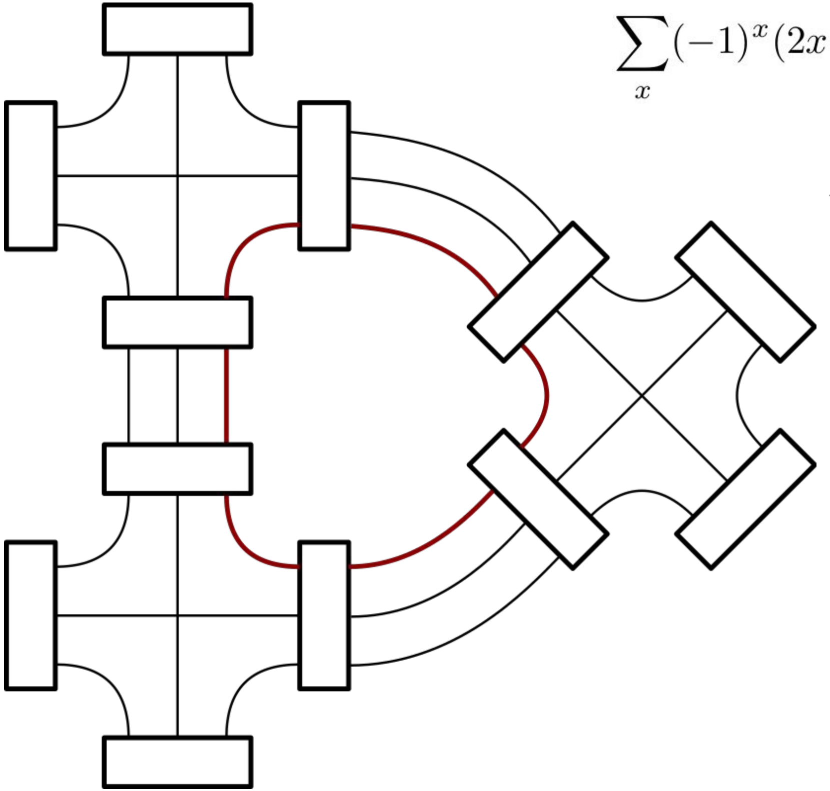

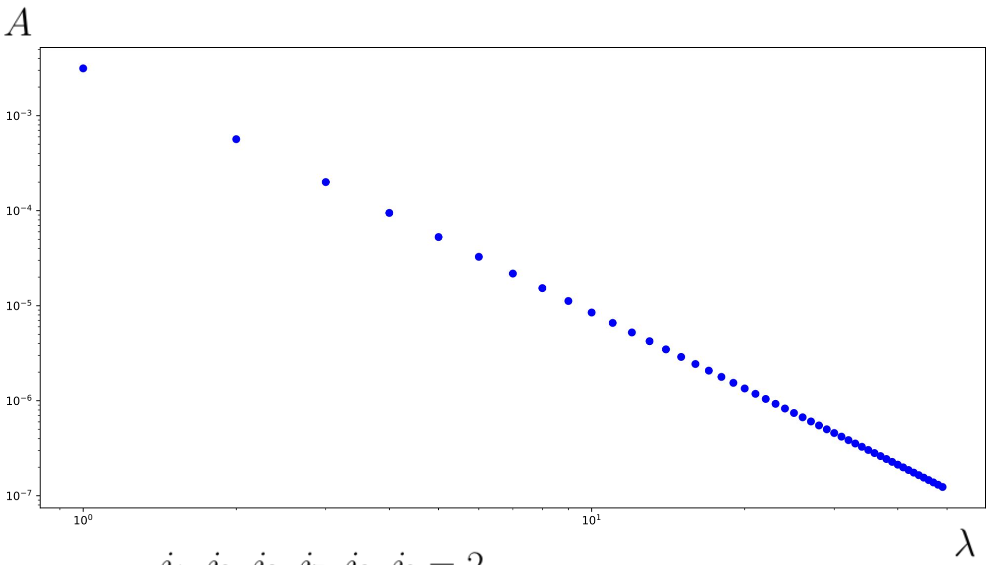

$$j_1, j_2, j_3, j_7, j_8, j_9 = 2$$
$$j_4, j_5, j_6 = 3$$

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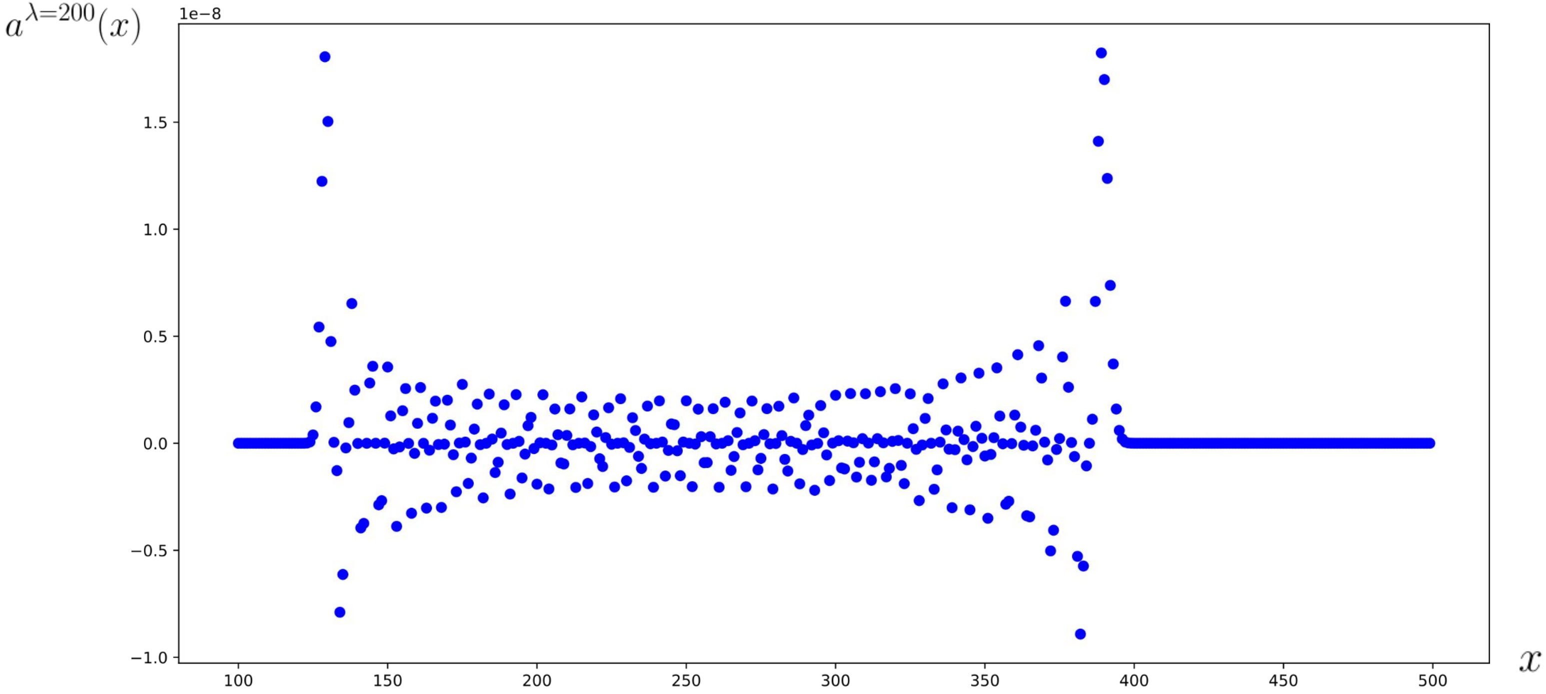

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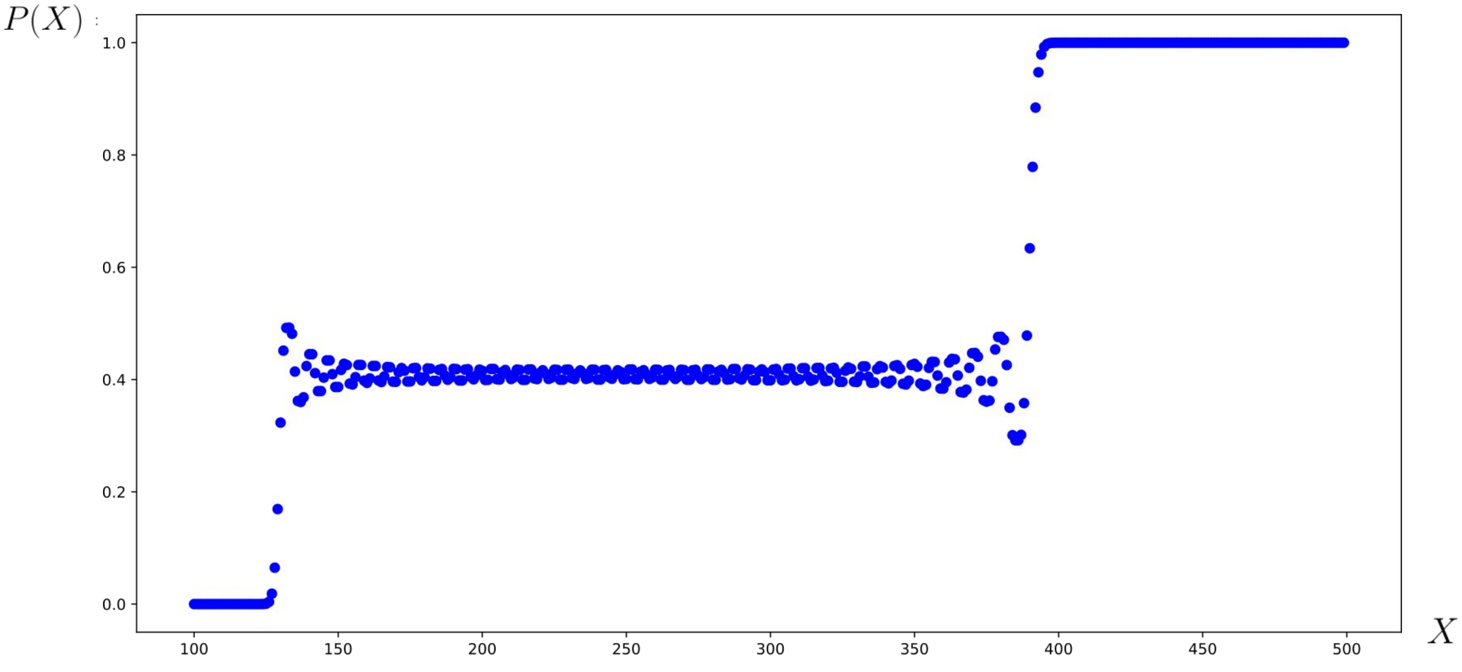
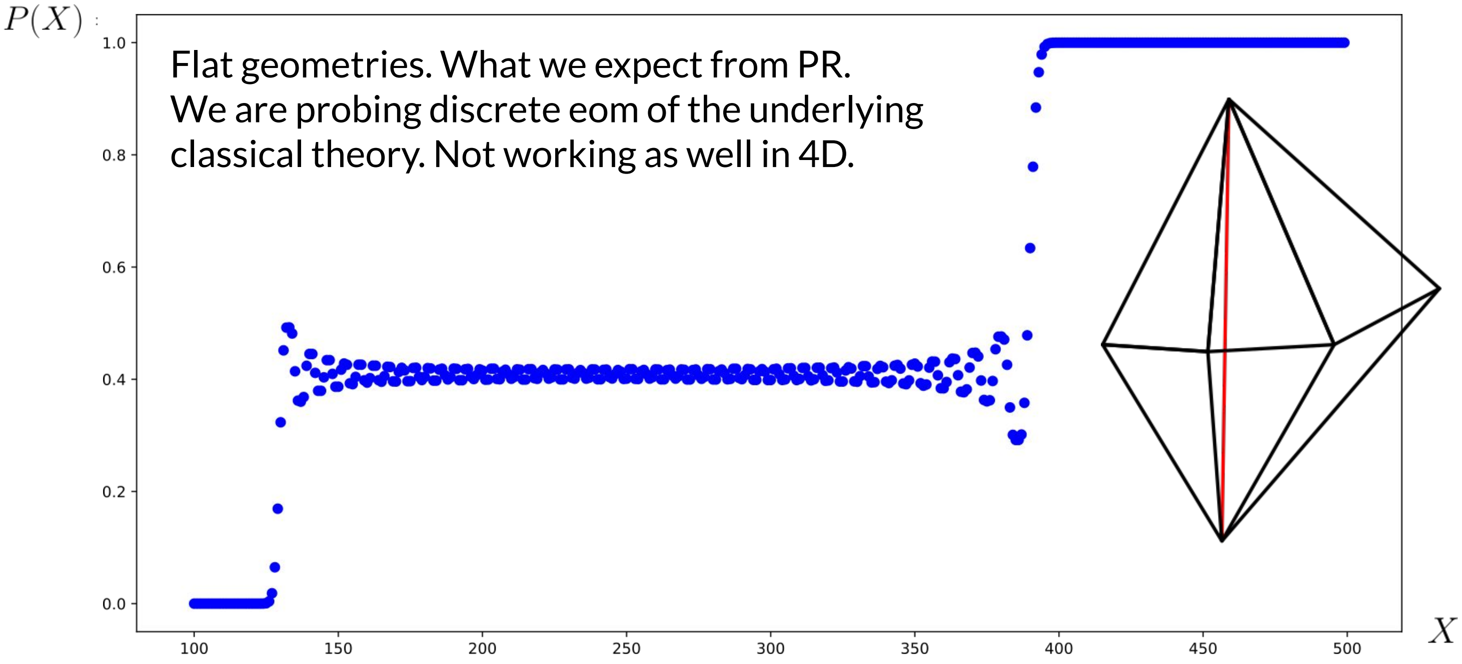
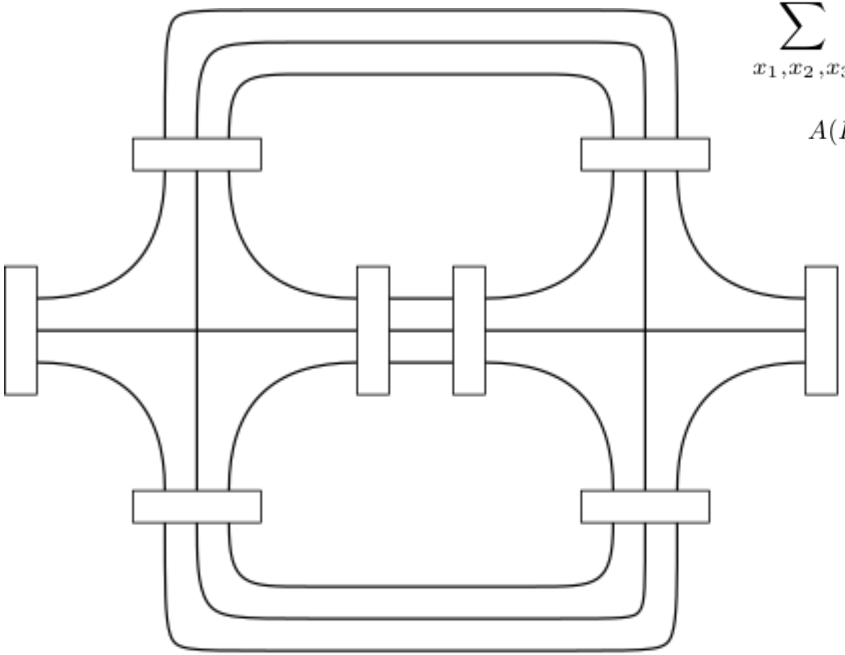


3D Gravity: The Ponzano-Regge model - 3 vertices

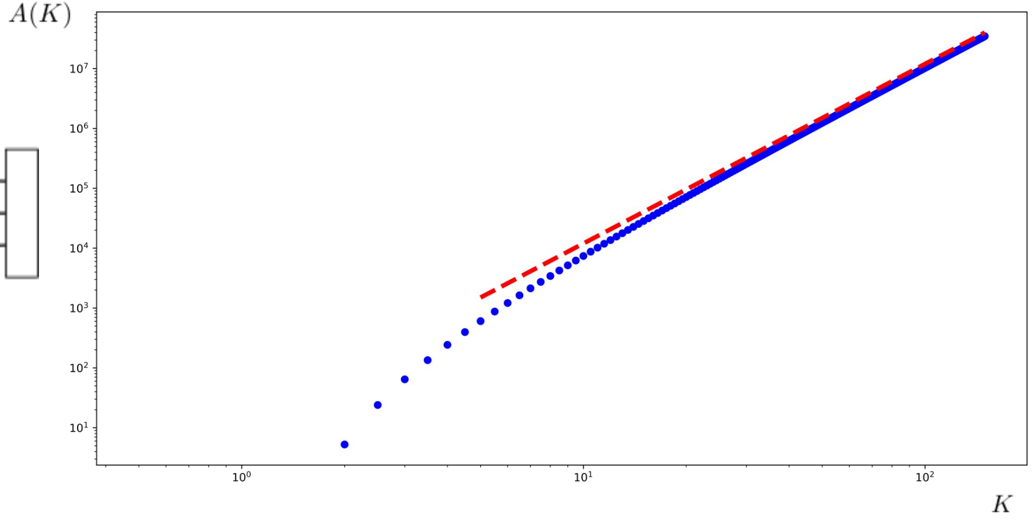
$$P(X) = \sum_{x}^{X} (-1)^x (2x+1) \begin{Bmatrix} j_5 & j_8 & x \\ j_9 & j_6 & j_1 \end{Bmatrix} \begin{Bmatrix} j_9 & j_6 & x \\ j_4 & j_7 & j_2 \end{Bmatrix} \begin{Bmatrix} j_4 & j_7 & x \\ j_8 & j_5 & j_3 \end{Bmatrix}$$



3D Gravity: The Ponzano-Regge model - The bubble



$$\sum_{x_1,x_2,x_3}^{K} (2x_1+1)(2x_2+1)(2x_3+1) \begin{Bmatrix} j_1 & j_2 & j_3 \\ x_1 & x_2 & x_3 \end{Bmatrix} \begin{Bmatrix} j_1 & j_2 & j_3 \\ x_1 & x_2 & x_3 \end{Bmatrix}$$



$$\delta(\mathbb{1}) = \sum_{x=0}^{K} (2x+1)^2 \approx K^3$$

Let's have a short break

4D Gravity: The EPRL model

LQG vertex with finite Immirzi parameter - Jonathan Engle, Etera Livine, Roberto Pereira, Carlo Rovelli
Covariant Loop Quantum Gravity - Carlo Rovelli, Francesca Vidotto

Transition amplitudes for Lorentzian 4D LQG (why is defined like this? - Maïté)

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Transition amplitudes for Lorentzian 4D LQG (why is defined like this? - Maïté)

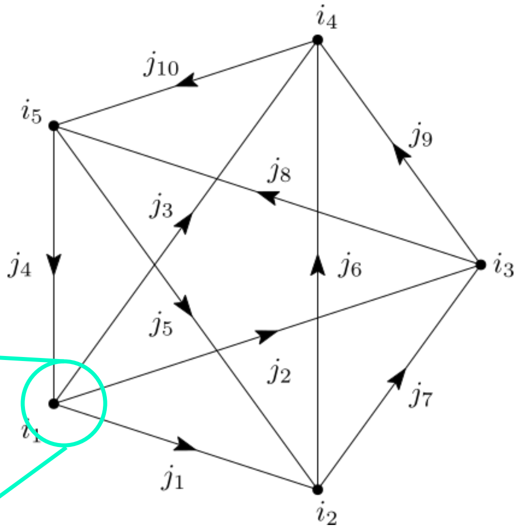
boundary state

Restrict to 4 valent spinnetworks (KKL extension)

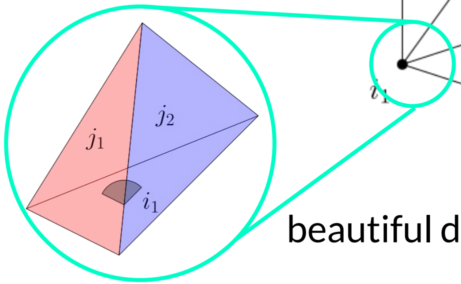
Spins as areas eigenvalues

Intertwiners in 4D = Quantum tetrahedra (closure)

(Quantum) triangulations of a 3D surface (twisted geometries)

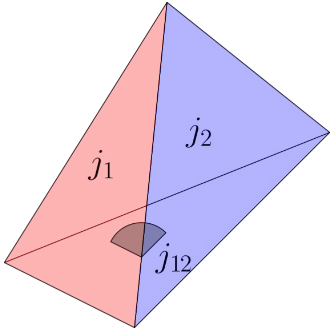


$$i = \begin{pmatrix} j_1 & j_2 & j_3 & j_4 \\ m_1 & m_2 & m_3 & m_4 \end{pmatrix}^{(j_{12})}$$



beautiful description given by Hal Haggard yesterday.

4 VALENT INTERTWINER = QUANTUM TETRAHEDRON

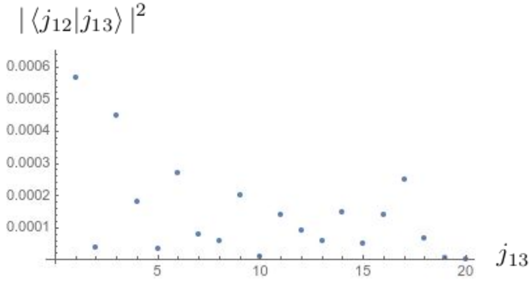


$$i = \begin{pmatrix} j_1 & j_2 & j_3 & j_4 \\ m_1 & m_2 & m_3 & m_4 \end{pmatrix}^{(j_{12})} = \sum_m (-1)^{j_{12}-m} \begin{pmatrix} j_1 & j_2 & j_{12} \\ m_1 & m_2 & m \end{pmatrix} \begin{pmatrix} j_{12} & j_3 & j_4 \\ -m & m_3 & m_4 \end{pmatrix}$$

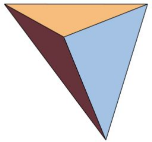
Areas: $J_i^2 \,|i\rangle = j_i(j_i+1)\,|i\rangle$

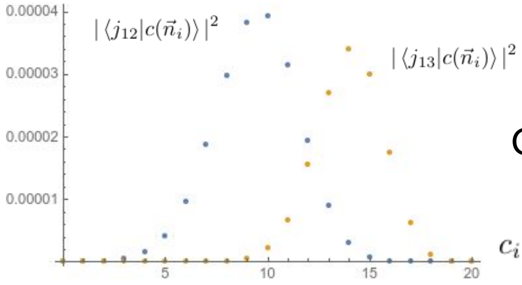
Closure: $|i\rangle \in \mathrm{Inv}\left(H^{(j_1)} \otimes H^{(j_2)} \otimes H^{(j_3)} \otimes H^{(j_4)}\right) \to \left(\vec{J}_1 + \vec{J}_2 + \vec{J}_3 + \vec{J}_4\right)|i\rangle = 0$

D. angle: $\vec{J}_1 \cdot \vec{J}_2 \,|i\rangle = \dfrac{j_{12}(j_{12}+1) - j_1(j_1+1) - j_2(j_2+1)}{2}\,|i\rangle$



Other angle: $\vec{J}_1 \cdot \vec{J}_3 \,|i\rangle = ?$ very spread!





Coherent intertwiner:

$$c_i(\vec{n_i}) = \begin{pmatrix} j_1 & j_2 & j_3 & j_4 \\ m_1 & m_2 & m_3 & m_4 \end{pmatrix}^{(j_{12})} \langle j_1 m_1 | j_1 \vec{n}_1\rangle \langle j_2 m_2 | j_2 \vec{n}_2\rangle \langle j_3 m_3 | j_3 \vec{n}_3\rangle \langle j_4 m_4 | j_4 \vec{n}_4\rangle$$

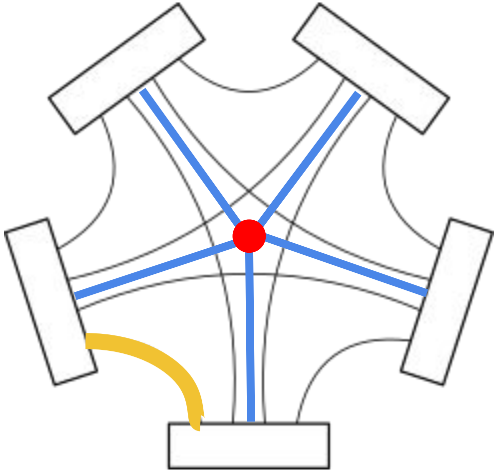
4D Gravity: The EPRL model

LQG vertex with finite Immirzi parameter - Engle, Livine, Pereira, Rovelli
Covariant Loop Quantum Gravity - Rovelli, Vidotto
Maïté's lectures at this school

Recipe to write any amplitude:

Choose a 2 complex dual to a 4D triangulation (vertices, edges, faces)

- 4simplex
- tetrahedron
- triangle



SL(2,C) in one slide

SL(2,C) Algebra

$$[L_i, L_j] = i\epsilon_{ijk}L_k$$
$$[L_i, K_j] = i\epsilon_{ijk}K_k$$
$$[K_i, K_j] = -i\epsilon_{ijk}L_k$$

Unit. Irreps: $\rho \in \mathbb{R} \quad k \in \mathbb{Z}/2$

$$(K^2 - L^2)\,|\rho,k;\ j,m\rangle = (\rho^2 - k^2 + 1)\,|\rho,k;\ j,m\rangle$$
$$\vec{K}\cdot\vec{L}\,|\rho,k;\ j,m\rangle = \rho k\,|\rho,k;\ j,m\rangle$$
$$L^2\,|\rho,k;\ j,m\rangle = j(j+1)\,|\rho,k;\ j,m\rangle$$
$$L_3\,|\rho,k;\ j,m\rangle = m\,|\rho,k;\ j,m\rangle$$

Group matrix elements in the irrep (ρ, k) $\quad \langle \rho,k;\ l,n|\,h\,|\rho,k;\ j,m\rangle = D^{\rho,k}_{lnjm}(h)$

$l, j \geq k$

Numerics need a parametrization - many - Cartan $\quad h = u e^{\frac{r}{2}\sigma_3} v^\dagger \qquad u, v \in SU(2),\ r \in \mathrm{R}^+$

Integration of functions on the group using the Haar measure $\quad \int dh = \int du dv \sinh(r) dr$

Matrix elements explicit formula

$$d^{(\rho,k)}_{jlm}(r) = (-1)^{j-l}\sqrt{\frac{(i\rho-j-1)!\,(j+i\rho)!}{(i\rho-l-1)!\,(l+i\rho)!}}\,\frac{\sqrt{(2j+1)(2l+1)}}{(j+l+1)!}\,e^{(i\rho-k-m-1)r}$$
$$\sqrt{(j+k)!(j-k)!(j+m)!(j-m)!(l+k)!(l-k)!(l+m)!(l-m)!}$$
$$\sum_{s,t}(-1)^{s+t}e^{-2tr}\frac{(k+s+m+t)!(j+l-k-m-s-t)!}{t!s!(j-k-s)!(j-m-s)!(k+m+s)!(l-k-t)!(l-m-t)!(k+m+t)!}$$
$${}_2F_1\left[\{l-i\rho+1, k+m+s+t+1\},\{j+l+2\};1-e^{-2r}\right]$$

$$D^{\rho,k}_{lnjm}(h) = \sum_o D^l_{no}(u)\,d^{\rho,k}_{ljo}(r)\,D^j_{om}(v^\dagger)$$

A Primer of Group Theory for Loop Quantum Gravity and Spin-foams - Pierre Martin-Dussaud

The Lorentz group and harmonic analysis - W. Rühl

4D Gravity: The EPRL model

LQG vertex with finite Immirzi parameter - Engle, Livine, Pereira, Rovelli
Covariant Loop Quantum Gravity - Rovelli, Vidotto
Maïté's lectures at this school

Impose simplicity constraints as a restriction on irreps

$$|j, m\rangle \xrightarrow[Y_\gamma]{} |\gamma j, j\,;\; j, m\rangle$$

γ Immirzi parameter

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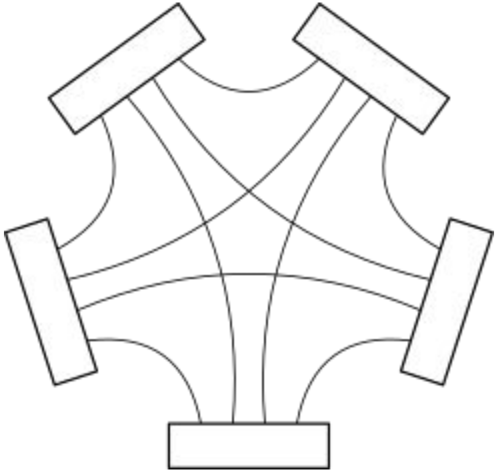

For each vertex:

For each edge one group element h_e

For each face an SU(2) spin j_f

and a matrix element $D^{\gamma j_f, j_f}_{j_f n\, j_f m}(h_s^{-1} h_t)$

$$D^{\gamma j_f, j_f}_{j_f n\, j_f m}(h_s^{-1} h_t) = \sum_{l=j_f}^{\infty} \sum_{o=-l}^{l} D^{\gamma j_f, j_f}_{j_f n\, l o}(h_s^{-1}) D^{\gamma j_f, j_f}_{l o\, j_f m}(h_t)$$

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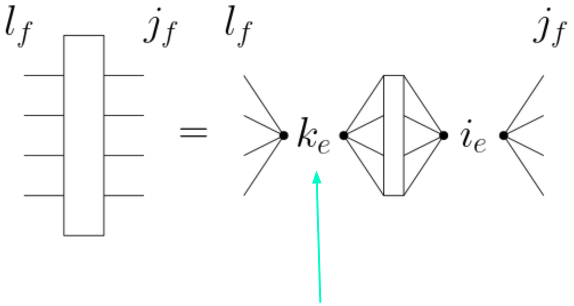
Perform the integrals over SL(2,C) using the Cartan decomposition:



[Boosting Wigner's nj-symbols - Simone Speziale](#)

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$$A_v(j_f, i_e) = \sum_{l_f, k_e} \left(\prod_e d_{k_e} B_4(j_f, l_f, i_e, k_e) \right) \{15j\}(l_f, k_e)$$

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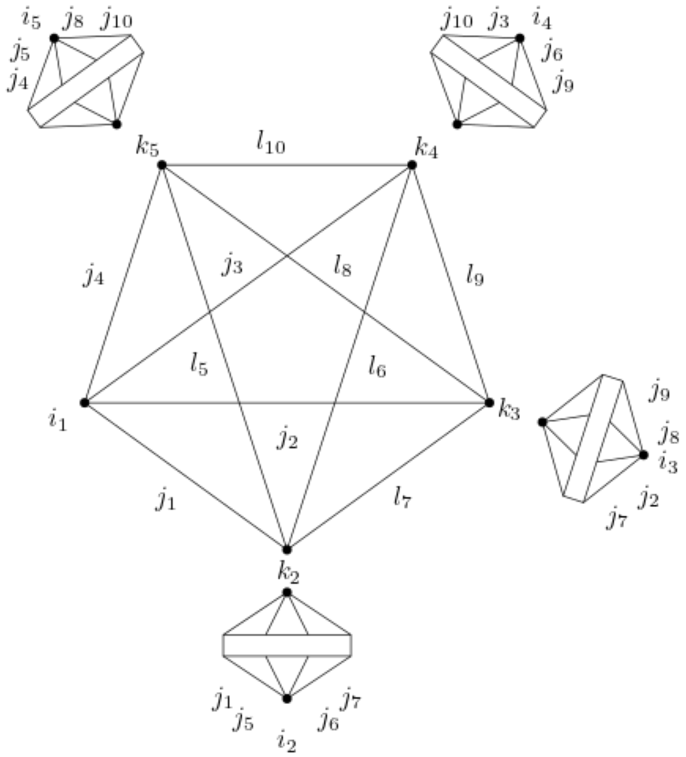

SL2CFOAM

sl2cfoam - P.D. and Giorgio Sarno

- Written in C
- SU(2) invariants WIGXJPF
- Arbitrary precision arithmetic
- parallelizable
- ✓ First attempt, good as long as it works
- ✗ largely unoptimized, slow

Fast and Accurate Evaluation of Wigner 3j, 6j, and 9j Symbols Using Prime Factorization and Multiword Integer Arithmetic - H. T. Johansson and C. Forssén

SL2CFOAM-NEXT

sl2cfoam-next - Francesco Gozzini
(the library is available on git, the paper will appear soon)

- Written in C
- SU(2) invariants WIGXJPF
- Arbitrary precision arithmetic
- Optimized to be efficient (boosters, data structures)
- HPC ready (OMP, MPI, GPU)
- Julia interface

SL2CFOAM-NEXT

sl2cfoam-next - Francesco Gozzini
(the library is available on git, the paper will appear soon)

User friendly!

```
using SL2Cfoam

# init SL2Cfoam library
Immirzi = 0.1;
folder = "/home/francesco/phd/src/data_sl2cfoam_next";
conf = SL2Cfoam.Config(VerbosityOff, NormalAccuracy, 100, 0);
SL2Cfoam.cinit(folder, Immirzi, conf);

# compute vertex tensor for increasing shells (j = 1, Δs = 0 → 4)
vs = [ vertex_compute(ones(10), s) for s in 0:4 ];

# print values value with intertwiners (0,0,0,0,0)
[ v.a[1, 1, 1, 1, 1] for v in vs ]
```

```
5-element Array{Float64,1}:
 5.992940521928846e-8
 8.77026353463005e-8
 1.0511153396261617e-7
 1.161678351516197e-7
 1.2344027926668653e-7
```

harder better faster

	j = 1	j = 3	j = 5	j = 7
0s:	1x	18x	900x	13000x
	j = 1	j = 2	j = 3	j = 4
1s:	24x	140x	1600x	12500x

pics stolen from Francesco's ILQGS seminar

EPRL VERTEX AMPLITUDE : DIVIDE AND CONQUER

$$A_v(j_f, i_e) = \sum_{l_f, k_e} \left(\prod_e d_{k_e} B_4(j_f, l_f, i_e, k_e) \right) \{15j\}(l_f, k_e)$$

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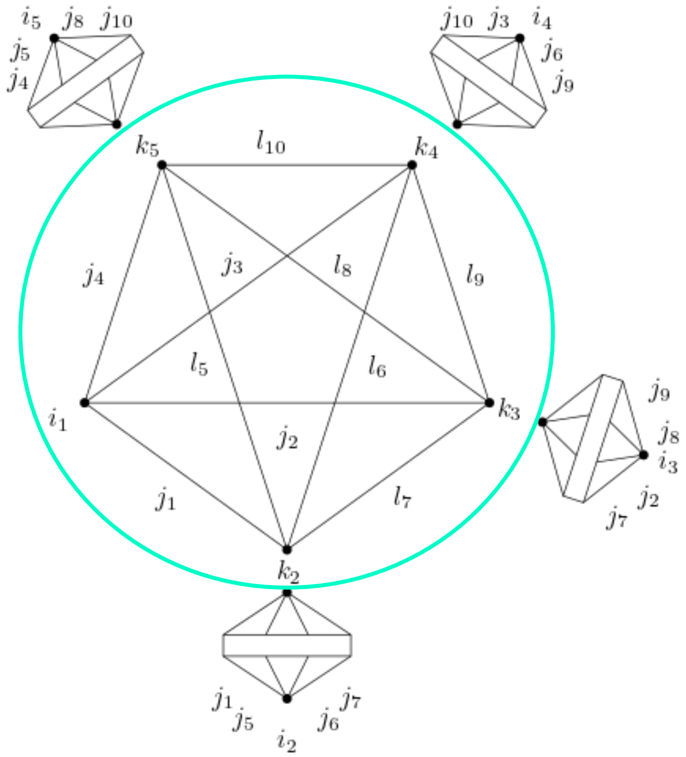
This is the simple part



[Fast and Accurate Evaluation of Wigner 3j, 6j, and 9j Symbols Using Prime Factorization and Multiword Integer Arithmetic - H. T. Johansson and C. Forssén]

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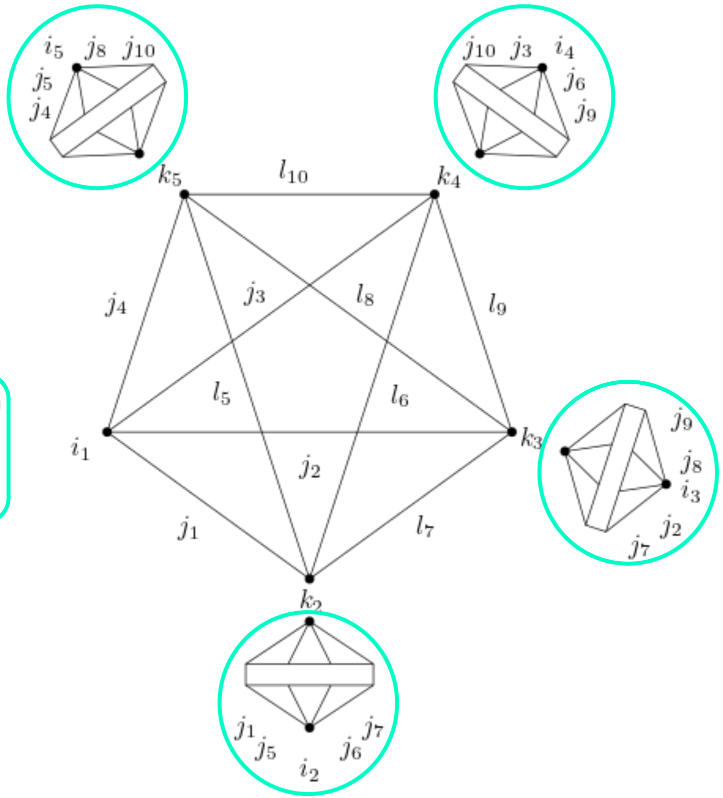
Relatively complicated



The booster functions

Encode the details of the model (imposition of the simplicity constraints)
Well understood semiclassical limit [Asymptotics of SL(2,C) coherent invariant tensors - P.D., Fanizza, Martin-Dussaud, Speziale]

$$B_4(l_f, j_f, i_e, k_e) \equiv \sum_{m_f} \begin{pmatrix} l_f \\ m_f \end{pmatrix}^{(i_e)} \left(\int_0^\infty d\mu(r) \prod_f d^{(\gamma j_f, j_f)}_{l_f j_f m_f}(r) \right) \begin{pmatrix} j_f \\ m_f \end{pmatrix}^{(k_e)}$$

We know the analytic result of the integral. Related to SL(2,C) Clebsch-Gordan coefficient.

[Boosting Wigner's nj-symbols - Simone Speziale]

Not efficient to implement numerically.

The booster functions

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Unbounded? Not really!

$$d^{(\rho,k)}_{jlm}(r) = (-1)^{j-l} \sqrt{\frac{(i\rho - j - 1)!\,(j + i\rho)!}{(i\rho - l - 1)!\,(l + i\rho)!}} \frac{\sqrt{(2j+1)(2l+1)}}{(j+l+1)!} e^{(i\rho - k - m - 1)r}$$
$$\sqrt{(j+k)!(j-k)!(j+m)!(j-m)!(l+k)!(l-k)!(l+m)!(l-m)!}$$
$$\sum_{s,t} (-1)^{s+t} e^{-2tr} \frac{(k+s+m+t)!(j+l-k-m-s-t)!}{t!s!(j-k-s)!(j-m-s)!(k+m+s)!(l-k-t)!(l-m-t)!(k+m+t)!}$$
$${}_2F_1\left[\{l - i\rho + 1, k+m+s+t+1\}, \{j+l+2\}; 1 - e^{-2r}\right]$$

The booster functions

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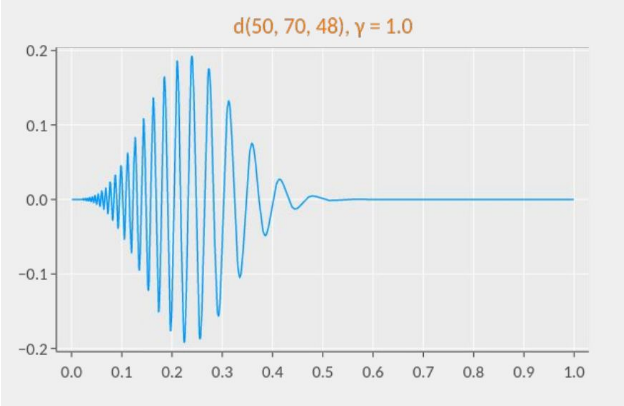
High frequency oscillations!



The result is in cancellations!

- Arbitrary precision routines (long double has not enough precision)
- Adaptive Gauss–Kronrod quadrature method for numerical integration (1D, sampling not known in advance, recycle points) - adapted from GSL source

The sums

sl2cfoam-next - Francesco Gozzini (to appear soon)

$$A_v(j_f, i_e) = \sum_{l_f, k_e} \left(\prod_e d_{k_e} B_4(j_f, l_f, i_e, k_e) \right) \{15j\}(l_f, k_e)$$

Store everything in multidimensional arrays

Sums thought as tensors contractions. Optimized routines (BLAS) instead of caveman loops

Same technique to contract vertices

The sums - The downside

$$A_v(j_f, i_e) = \sum_{l_f, k_e} \left(\prod_e d_{k_e} B_4(j_f, l_f, i_e, k_e) \right) \{15j\}(l_f, k_e)$$

$$D^{\gamma j_f, j_f}_{j_f n\, j_f m}(h_s^{-1} h_t) = \sum_{l=j_f}^{\infty} \sum_{o=-l}^{l} D^{\gamma j_f, j_f}_{j_f n\, l o}(h_s^{-1}) D^{\gamma j_f, j_f}_{l o\, j_f m}(h_t)$$

To perform numerical computation we need to cutoff these sums

$$\sum_{l=j_f}^{\infty} \approx \sum_{l=j_f}^{j_f+\Delta s}$$

THE SUMS - THE DOWNSIDE

$$A_v(j_f, i_e) = \sum_{l_f=j_f}^{j_f+\Delta s} \sum_{k_e} \left(\prod_e d_{k_e} B_4(j_f, l_f, i_e, k_e) \right) \{15j\}(l_f, k_e)$$

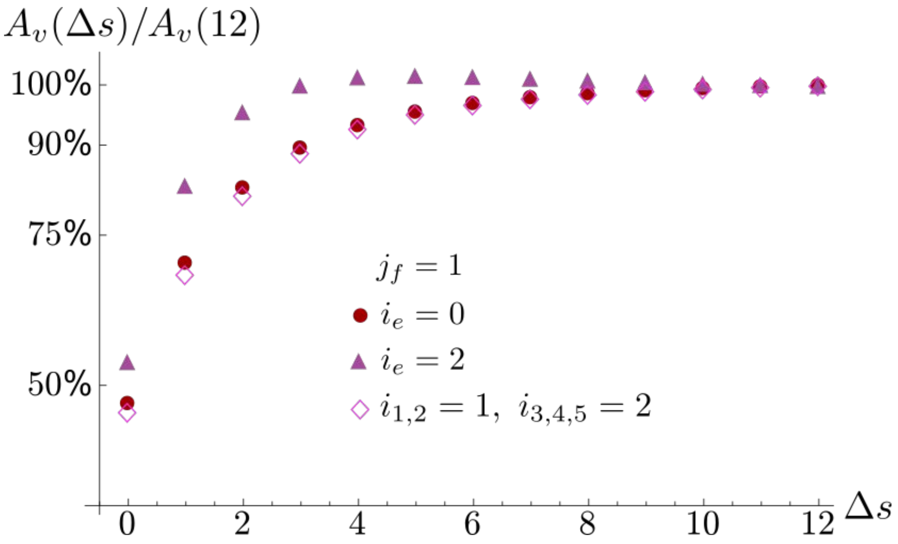
6 sums. We decided to impose uniformly the cutoff

- We know the sums are convergent
- Slang: Δs = shells
- Good approximation (eventually)
- Empirical error of the truncation (bad)
- depends on everything

Pietropaolo Frisoni, Carlo Rovelli, Francesco Gozzini, P. D.

Strategy: as much as possible! Error?

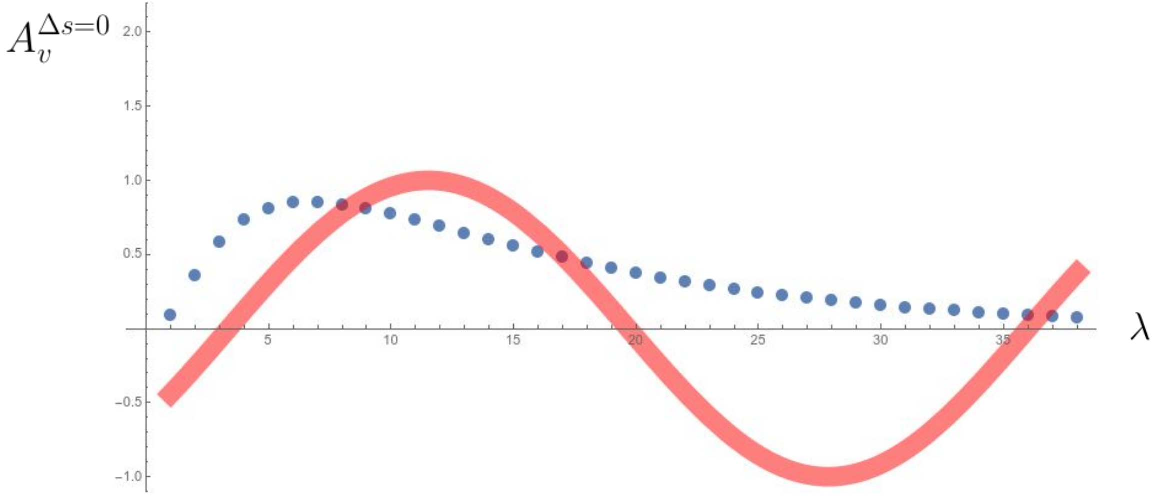
Regularization and finiteness of the Lorentzian LQG vertices - Jonathan Engle, Roberto Pereira



Applications - Test the asymptotic formula

Similar to P.R. EPRL amplitude with bdr coherent states peaked on the bdr of a Lorentzian 4-simplex

- power law behavior
- oscillation frequency equal to Regge Action of the Lorentzian 4-simplex



Numerical study of the Lorentzian EPRL spin foam amplitude - Pietro Dona, Marco Fanizza, Giorgio Sarno, Simone Speziale
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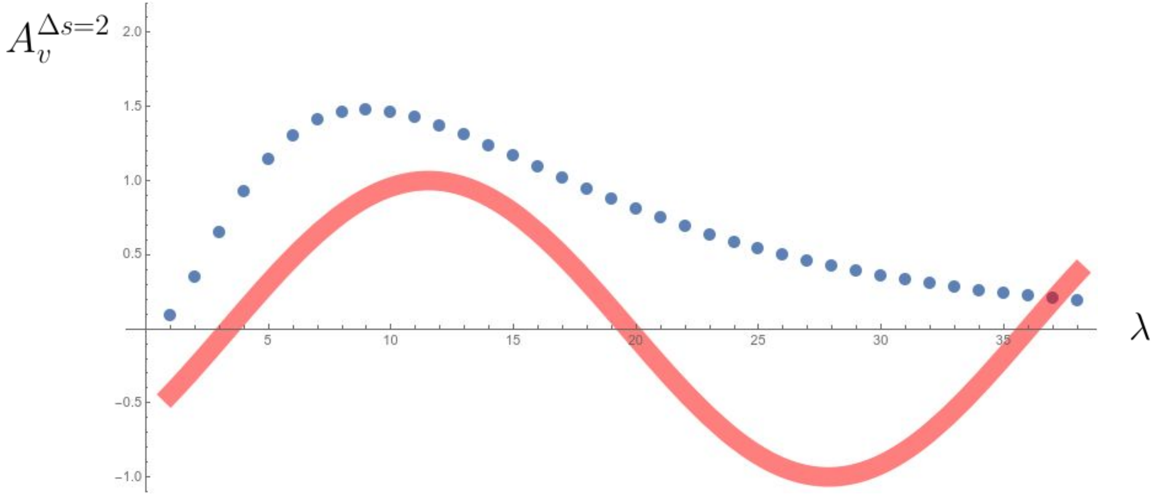

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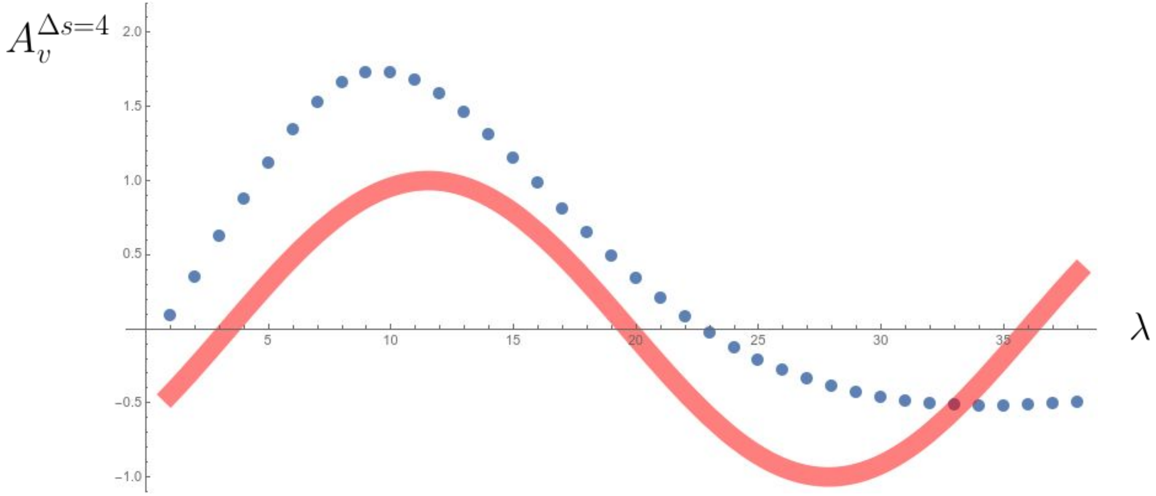

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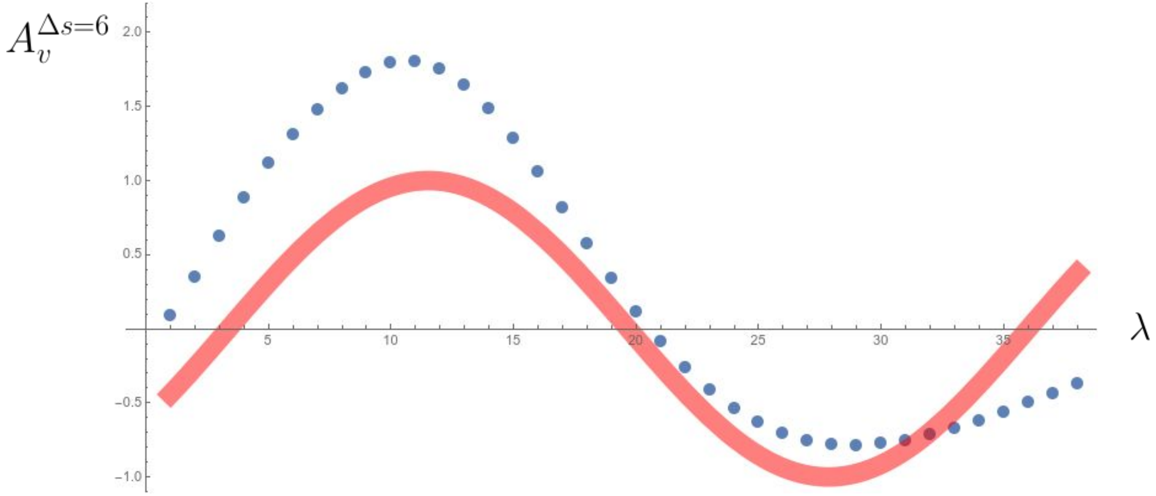
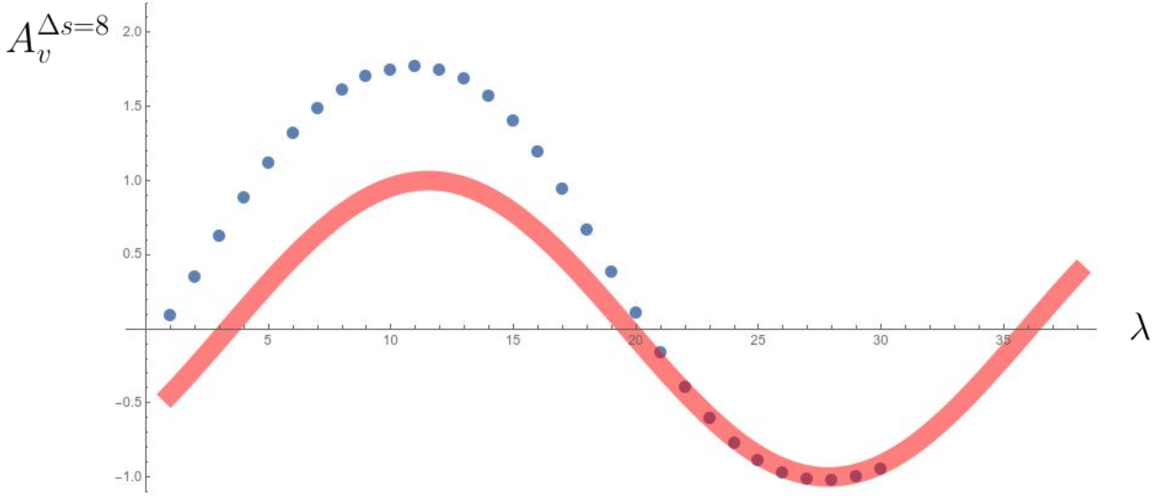


Numerical study of the Lorentzian EPRL spin foam amplitude - Pietro Dona, Marco Fanizza, Giorgio Sarno, Simone Speziale
sl2cfoam-next - Francesco Gozzini (to appear soon)

Applications - Test the asymptotic formula

Similar to P.R. EPRL amplitude with bdr coherent states peaked on the bdr of a Lorentzian 4-simplex

- power law behavior
- oscillation frequency equal to Regge Action of the Lorentzian 4-simplex



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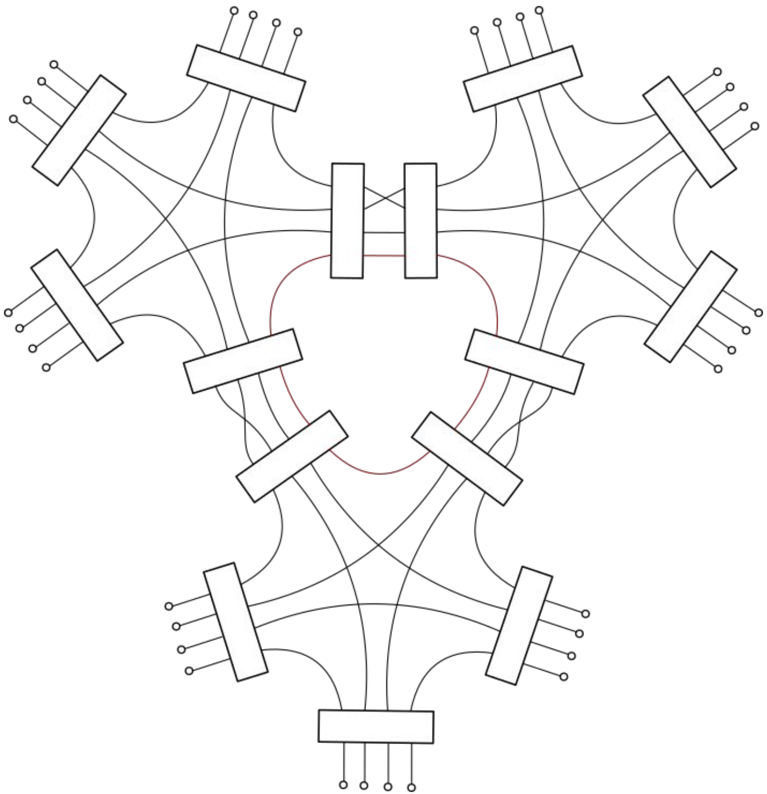
Applications - 3 Vertex amplitude - 1 internal face



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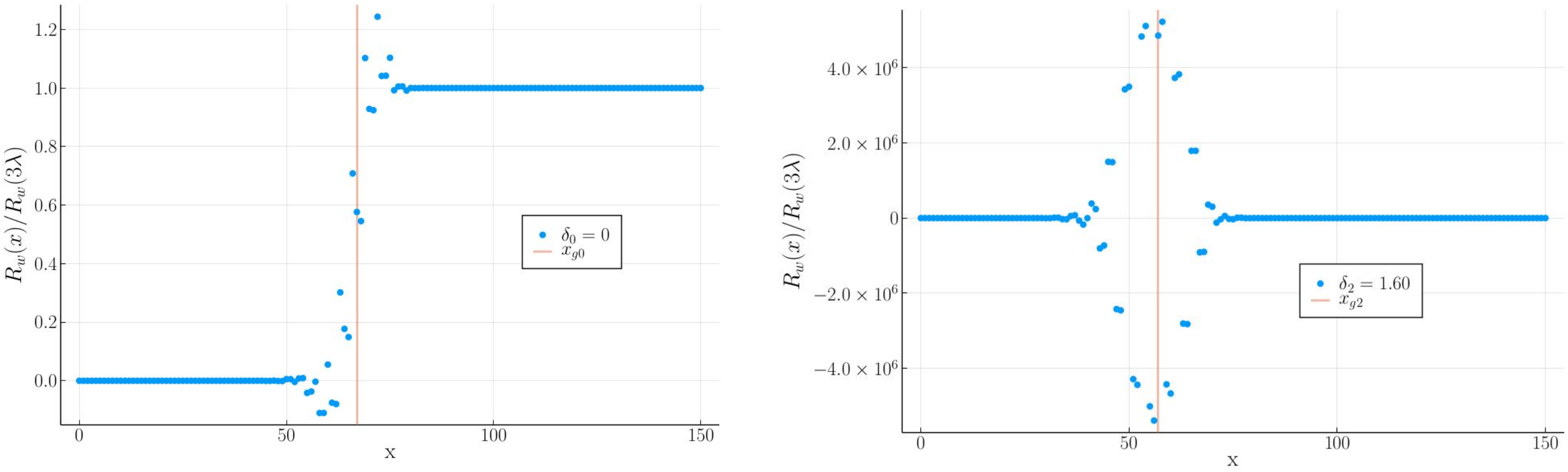

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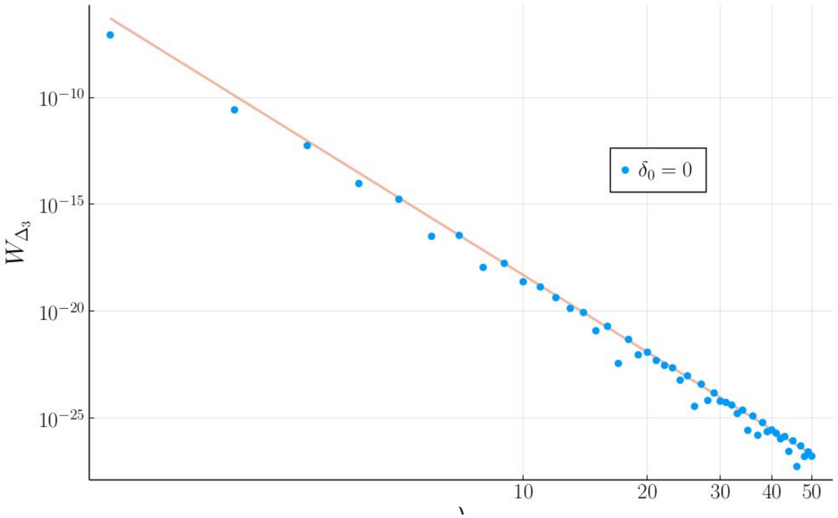
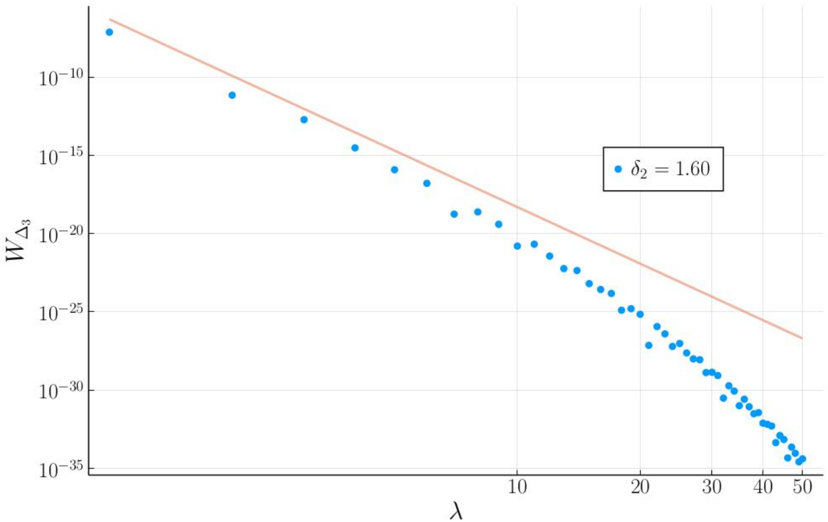
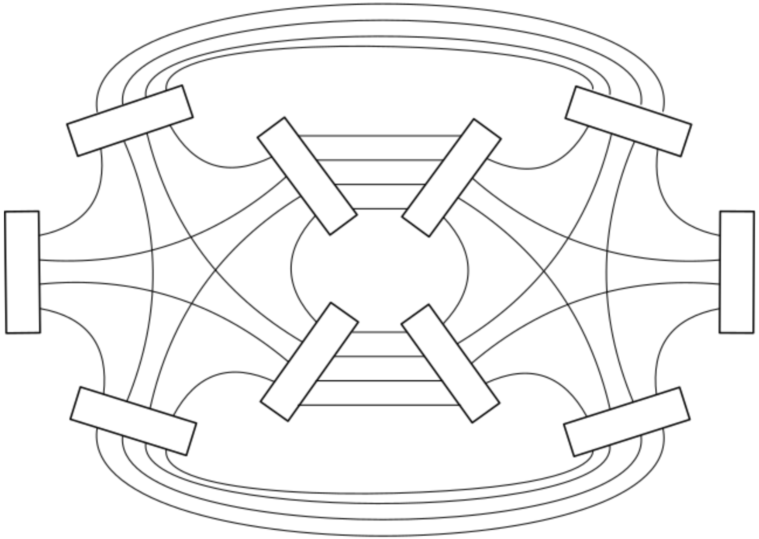




Applications - IR (large-volume) divergences

Two results in the literature, IR melonic divergences, necessary step for renormalization

- bubbles are divergent — (the same kind in Valentin Bonzom's lecture)
- bounded from below - $\log K$ — Self-Energy of the Lorentzian EPRL-FK Spin Foam Model of Quantum Gravity - Aldo Riello
- bounded from above - K^9 — Infrared divergences in the EPRL-FK Spin Foam model - P.D.
- numerical estimation is in progress — w.i.p. Francesco Gozzini, **Pietropaolo Frisoni**, Carlo Rovelli, Francesca Vidotto



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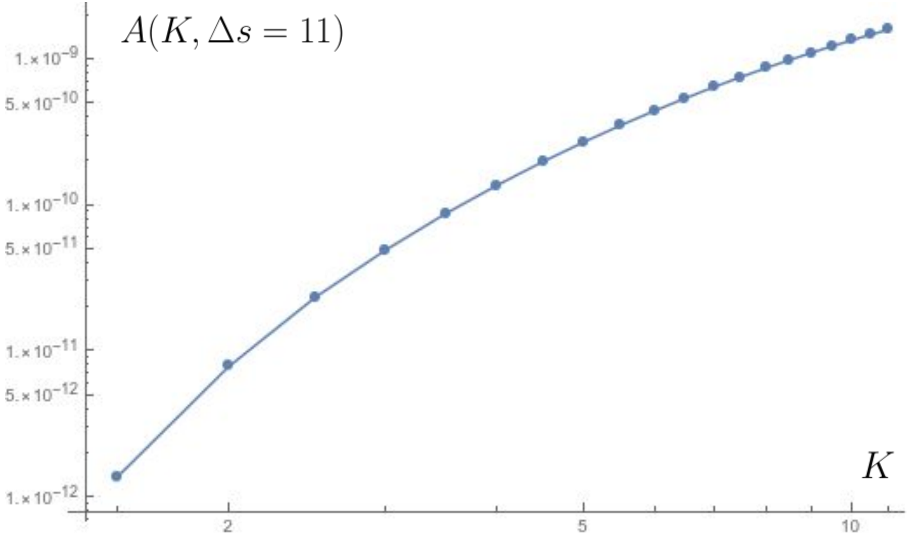

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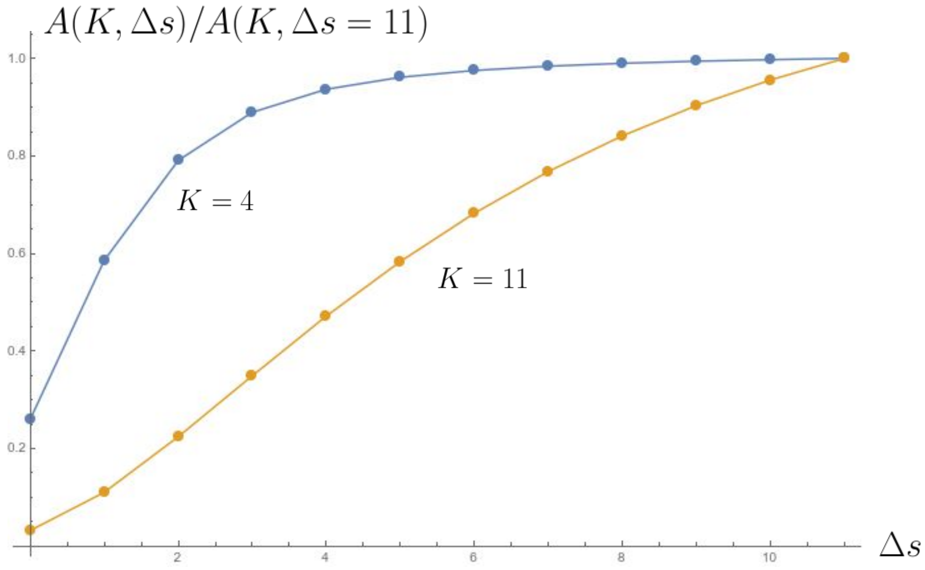

Conclusion

Doing computations with the Ponzano Regge model is an excellent way to play with spin foams on your laptop. (more minor technical difficulties of 4D, shares formal aspects, extension is not straightforward).

We have a well-written, optimized, parallelizable, user-friendly code (sl2cfoam-next) that allows computing EPRL Lorentzian spin foam amplitudes. (Elementary ones on your laptop, to do serious computations, HPC is needed - our resources are orders of magnitudes smaller than LQCD guys ones)

Field in constant development. We change idea very quickly! We should not be afraid to.

If you want to contribute you are very welcome to fork the repository and have fun!

- Implement Euclidean EPRL model (bounded sums) (P.D. , Francesco Gozzini, Alessandro Nicotra)
- Extrapolation in the internal spin summation (truncation control) (Pietropaolo Frisoni, Francesco Gozzini)
- Amplitudes representing physical processes (Carlo Rovelli, Farshid Soltani and Francesca Vidotto @Western U. on bouncing black holes)