## Exercise 1 - Electric field induced by the variable magnetic field

A solenoid of circular cross-section, with radius $R$, length $l$, made up of $N$ turns, is traversed by an alternating current $I = I_0 \cos \omega t$. Determine the electric field induced outside the solenoid.

The magnetic field of a solenoid vanishes outside the solenoid and is uniform inside the solenoid with the direction of the axis of the solenoid. In cylindrical coordinates (with the spires of the solenoid in the $r, \phi$ plane), the magnetic field inside the solenoid is

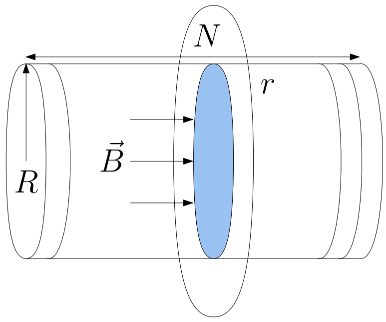
$$\vec{B} = \frac{\mu_0 N I}{\ell} \vec{e}_z = \frac{\mu_0 N I_0}{\ell} \cos(\omega t) \vec{e}_z \ . \tag{1}$$

Consider now a circular path of radius $r$ on the plane of a coil of the solenoid. The induced electric field is radial $\vec{E} = E\vec{e}_\phi$, and the Faraday's law relates

$$\oint \vec{E} \cdot \mathrm{d}\vec{\ell} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} \ . \tag{2}$$

The integral of the electric field is, in all the cases

$$\oint \vec{E} \cdot \mathrm{d}\vec{\ell} = E 2\pi r \ . \tag{3}$$




Outside the solenoid $r > R$, the flux of the magnetic field through the surface enclosed by the circular path is always the same and depends only on the magnetic field and the area of a spire

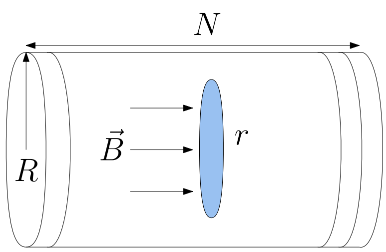
$$\Phi_{r>R} = \int \vec{B} \cdot \mathrm{d}\vec{S} = \frac{\mu_0 N I_0}{\ell} \cos(\omega t) \pi R^2 \qquad \text{and} \qquad \frac{\mathrm{d}\Phi_{r>R}}{\mathrm{d}t} = -\omega \pi R^2 \frac{\mu_0 N I_0}{\ell} \sin \omega t \ . \tag{4}$$

Therefore,

$$E_{r>R} 2\pi r = -\omega \pi R^2 \frac{\mu_0 N I_0}{\ell} \sin(\omega t) \qquad \rightarrow \qquad E_{r>R} = -\frac{1}{2r} \omega R^2 \frac{\mu_0 N I_0}{\ell} \sin(\omega t) \ . \tag{5}$$

The electric field decreases as $1/r$ for $r > R$. The electric field also varies over time. Therefore, it will induce a magnetic field, which will induce an electric field, and so on. Every single time, the intensity will be multiplied by a factor $\omega$. Therefore, for low frequency, this is negligible. At high frequencies, this will become important and will be fundamental to describing electromagnetic waves.

Calculate the electric field induced inside the solenoid.



Inside the solenoid $r < R$, the flux of the magnetic field depends on the portion of the magnetic field contained in the circular path

$$\Phi_{r<R} = \int \vec{B} \cdot \mathrm{d}\vec{S} = \mu_0 N I_0 \cos\omega t \pi r^2 \ . \tag{6}$$

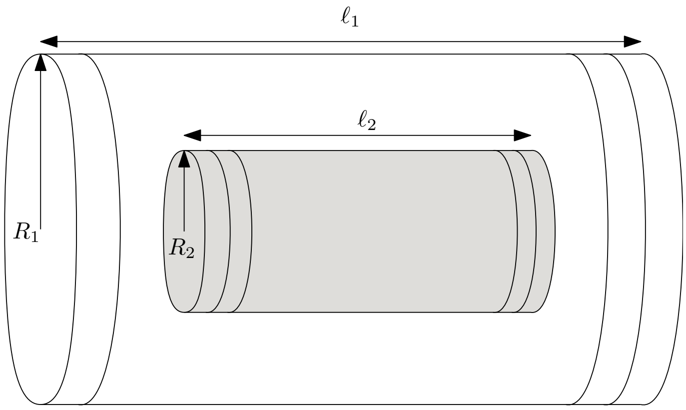
The induced electric field is

$$E_{r<R} 2\pi r = -\omega \pi r^2 \mu_0 N I_0 \sin\omega t \qquad \to \qquad E_{r<R} = -\frac{1}{2}\omega r \mu_0 N I_0 \sin\omega t \ . \tag{7}$$

Notice how the electric field is 0 at the center of the solenoid, increases linearly in $r$, and is orthogonal to $\vec{B}$.

## Exercise 2 - Mutual inductance — transformer

Consider a solenoid of circular cross-section, with radius $R_1$, length $\ell_1$, made up of $N_1$ turns. Inside it, a second solenoid with radius $R_2$, length $\ell_2$, and made up of $N_2$ turns are placed. Calculate the mutual inductance, $M$, between the two solenoids using the approximation of infinite solenoids.




For an infinite solenoid, the magnetic field $B$ inside is given by:

$$B = \mu_0 \frac{N}{\ell} I \ . \tag{8}$$

The total flux through the two solenoids (1 and 2) is

$$\Phi_1 = L_1 I_1 + M I_2 \ , \tag{9}$$
$$\Phi_2 = L_2 I_2 + M I_1 \ . \tag{10}$$

Where we used the fact that the mutual inductance $M = M_{12} = M_{21}$. The self-inductance $L_1$ is simple to compute since

$$\Phi_{1\to 1} = B_1 N_1 A_1 = \mu_0 \frac{N_1^2}{\ell_1} I_1 \pi R_1^2 = L_1 I_1 \ . \tag{11}$$

Similarly, we can compute $L_2$.

$$L_1 = \mu_0 \frac{N_1^2}{\ell_1} \pi R_1^2 \ , \qquad L_2 = \mu_0 \frac{N_2^2}{\ell_2} \pi R_2^2 \ . \tag{12}$$

To compute the mutual inductance, we can compute the flux through the solenoid at 2 due to the solenoid's magnetic field at 1

$$\Phi_{1\to 2} = M I_1 = B_1 N_2 A_2 = \mu_0 \frac{N_1 N_2}{\ell_1} I_1 \pi R_2^2 \ . \tag{13}$$

We deduce that

$$M = \mu_0 \frac{N_1 N_2}{\ell_1} \pi R_2^2 \tag{14}$$

Express $M$ in the case where $\ell_2 \to \ell_1$ and $R_2 \to R_1$, but with $N_1 \neq N_2$.

In the limit $\ell_1 \to \ell$, $\ell_2 \to \ell$ and $R_1 \to R$, $R_2 \to R$

$$M = \mu_0 \frac{N_1 N_2}{\ell} \pi R^2 = \sqrt{L_1 L_2} \tag{15}$$

which can be recognized as the square root of the product of the two self-inductances.

Recall that insofar as the wire resistances are negligible, the voltages across the solenoids are expressed (in the receiver convention):

$$U_1(t) = L_1 \frac{dI_1(t)}{dt} + M \frac{dI_2(t)}{dt} \tag{16}$$

$$U_2(t) = L_2 \frac{dI_2(t)}{dt} + M \frac{dI_1(t)}{dt} \tag{17}$$

Considering the case we studied in the previous question, calculate the ratio $U_2(t)/U_1(t)$ as a function of $N_1$ and $N_2$. Do you see an interesting application?

$$U_1(t) = \mu_0 \frac{N_1^2}{\ell} \pi R^2 \frac{dI_1(t)}{dt} + \mu_0 \frac{N_1 N_2}{\ell} \pi R^2 \frac{dI_2(t)}{dt} == \mu_0 \frac{N_1}{\ell} \pi R^2 \left( N_1 \frac{dI_1(t)}{dt} + N_2 \frac{dI_2(t)}{dt} \right) , \tag{18}$$

and

$$U_2(t) = \mu_0 \frac{N_2^2}{\ell} \pi R^2 \frac{dI_2(t)}{dt} + \mu_0 \frac{N_1 N_2}{\ell} \pi R^2 \frac{dI_1(t)}{dt} == \mu_0 \frac{N_2}{\ell} \pi R^2 \left( N_1 \frac{dI_1(t)}{dt} + N_2 \frac{dI_2(t)}{dt} \right) . \tag{19}$$

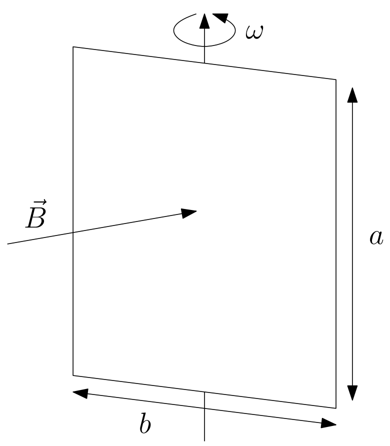
Therefore, the ratio depends on the ratio of the number of coils

$$\frac{U_2}{U_1} = \frac{\mu_0 \frac{N_2}{\ell} \pi R^2}{\mu_0 \frac{N_1}{\ell} \pi R^2} = \frac{N_2}{N_1} . \tag{20}$$

This is the mechanism at the electric transformer's base to convert the circuit's tension. https://en.wikipedia.org/wiki/Transformer

## Exercise 3 - Generator — Rotating Frame

A flat, rectangular, and non-deformable coil, with sides $a = 20cm$, $b = 10cm$, is made of a cylindrical conductor with a diameter of $d = 1mm$, and resistivity $\rho = 1.6 \times 10^{-8} \Omega m$. It rotates at a frequency of 600 revolutions per minute around a vertical axis located in the plane of the coil. The coil is placed in a magnetic field of intensity $B = 1T$, perpendicular to the axis of rotation (figure). What is the expression for the current flowing in the coil? Calculate its effective value.



Suppose that the magnetic field is in direction $\vec{e}_x$

$$\vec{B} = B\vec{e}_x \ . \tag{21}$$

We can use cylindrical coordinates to describe the motion of the coil. We assume that the coil is on the plane orthogonal to the vector

$$\vec{S} = S\vec{e}_r = S\left(\cos\phi(t)\vec{e}_x + \sin\phi(t)\vec{e}_y\right) \ , \tag{22}$$

where $\phi(t) = \omega t$, and $S = ab$. The flux of the magnetic field through the spire is

$$\Phi = \vec{B}\cdot\vec{S} = Bab\vec{e}_x \cdot (\cos\phi(t)\vec{e}_x + \sin\phi(t)\vec{e}_y) = Bab\cos\phi(t) \ . \tag{23}$$

The tension induced in the spire due to Faraday's law

$$\Delta V = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} = Bab\omega\sin\omega t \ . \tag{24}$$

The current induced in the spire is

$$I = \frac{\Delta V}{R} = \frac{Bab\omega}{R}\sin\omega t \ , \tag{25}$$

where the resistance is

$$R = \rho\frac{length}{section} = \rho\frac{2(a+b)}{\pi(\frac{d}{2})^2} \tag{26}$$

The effective value of the current is

$$I_{rms} = \sqrt{\frac{\omega}{2\pi}\int_0^{\frac{\omega}{2\pi}} I^2(t)} = \frac{1}{\sqrt{2}}\frac{Bab\omega}{R} \tag{27}$$

Describe the mechanical action of $\vec{B}$ on the loop.

The four coil segments have an (induced) current that generates a (Laplace) force when interacting with the magnetic field. We are interested in computing the momentum of the forces with respect to the rotation axis. The forces on the top and bottom segments of the coil have zero momentum as they are parallel to the rotation axis. The forces on the two sides are

$$\vec{F}_l = Ia\vec{e}_z \times \vec{B} == IaB\vec{e}_y \ , \qquad \vec{F}_r = Ia(-\vec{e}_z)\times\vec{B} = -IaB\vec{e}_y \ . \tag{28}$$

The side vector to the left side of the coil is $\vec{v} = -\sin\phi\vec{e}_x + \cos\phi\vec{e}_y$, and the one to the right side of the coil is $-\vec{v}$. The total momentum of the forces is

$$\vec{M} = \frac{b}{2}\vec{v}\times\vec{F}_l - \frac{b}{2}\vec{v}\times\vec{F}_r = b\vec{v}\times\vec{F}_l = -IabB\sin\phi\vec{e}_x\times\vec{e}_y = -IabB\sin\phi\vec{e}_z \tag{29}$$

The momentum of the forces opposes the rotation of the coil.