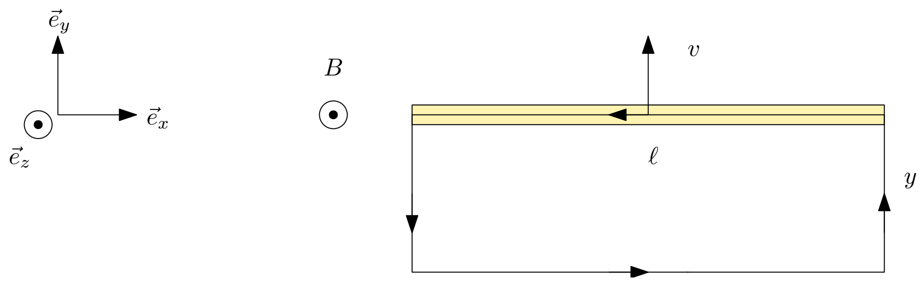
Exercise 1 - Conductor in a Uniform Magnetic Field

A metallic rod of length $\ell = 1.5m$ is placed in a uniform, constant magnetic field $B = 0.5T$. The rod is perpendicular to $\vec{B}$. It moves with a constant speed $v = 4m/s$ in a direction perpendicular to both $\vec{B}$ and the rod. Calculate the electric potential difference between the ends of the rod.



We set $\vec{B} = B\vec{e}_z$, and the rod spans the $\vec{e}_x$ direction and moves in the $\vec{e}_y$ direction. Faraday's Law relates the circulation of the electric field to the variation of the magnetic flux

$$\oint_{\partial S} \vec{E} \cdot \mathrm{d}\vec{\ell} = -\frac{\mathrm{d}}{\mathrm{d}t}\Phi_B = -\frac{\mathrm{d}}{\mathrm{d}t}\oint_S \vec{B} \cdot \mathrm{d}\vec{S} \,. \tag{1}$$

We consider the surface S to be a rectangle with the rod as one of its sides. The induction electric field is zero everywhere but on the rod itself. The left hand side of (1) is

$$\oint_{\partial S} \vec{E} \cdot \mathrm{d}\vec{\ell} = \int_0^\ell \vec{E}\mathrm{d}x = -\Delta V \,. \tag{2}$$

The magnetic field flux across the surface S is

$$\Phi_B = \oint_S \vec{B} \cdot \mathrm{d}\vec{S} = B\ell y \qquad \rightarrow \qquad -\frac{\mathrm{d}}{\mathrm{d}t}\Phi_B = -B\ell\frac{\mathrm{d}y}{\mathrm{d}t} = -B\ell v \,. \tag{3}$$

Where we used $\frac{\mathrm{d}y}{\mathrm{d}t} = v$. From Faraday's Law, we conclude

$$\Delta V = B\ell v \,. \tag{4}$$

Exercise 2 - Coil in a Time-Varying Uniform Magnetic Field

A coil with a radius $r = 4cm$, consisting of $N = 500$ turns, is placed in a uniform magnetic field that varies over time according to the law $B(t) = at + bt^4$. The coil is perpendicular to the magnetic field and is connected to a resistor $R = 600\Omega$. The resistance of the coil is neglected. Determine the induced electromotive force in the coil.

We directly apply the Faraday's law and get the electromotive force (emf) :

$$e = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -NS\frac{\mathrm{d}B}{\mathrm{d}t} = -NS\left(a + 4bt^3\right) \tag{5}$$

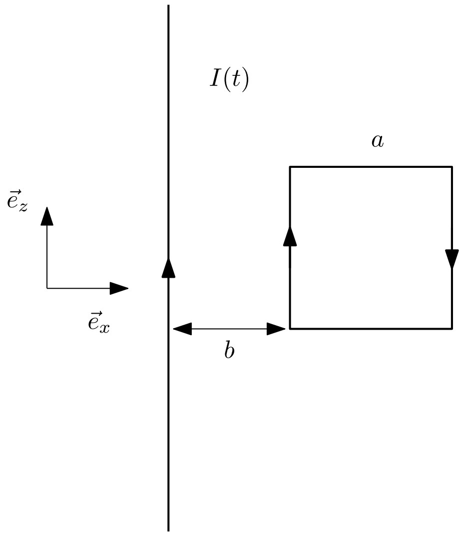
What is the current passing through the resistor at $t = 5s$? Use $a = 1.2 \times 10^{-2}Ts^{-1}$, and $b = 3 \times 10^{-5}Ts^{-4}$.

The current across the resistor after 5 seconds is

$$I = \frac{e}{R} = \frac{NS\left(a + 4bt^3\right)}{R} = 1.13 \cdot 10^{-4}\mathrm{A} \,. \tag{6}$$

Exercise 3 - Moving Coil

An infinite straight wire, carrying a constant current I, is located in the plane of a square conductive frame with side a (plane xz) at a distance b (see figure). The frame moves away from the wire with a constant speed v, orthogonal to the current, and in the plane of the frame (direction $\vec{e}_x$). Calculate the induced electromotive force (emf) in the frame due to the magnetic field produced by the infinite wire as a function of the distance b between the wire and the frame.



We computed the flux through the square coil in the TD6.

$$\Phi = \int_b^{a+b} \frac{\mu_0 I a}{2\pi} \frac{\mathrm{d}x}{x} = \frac{\mu_0 I a}{2\pi} \log \frac{b+a}{b} \tag{7}$$

To recap the calculation, the magnetic field generated by the wire on the plane of the square coil is

$$\vec{B} = \frac{\mu_0 I}{2\pi} \frac{1}{x} \vec{e}_y \tag{8}$$

To answer your questions from the last TD, we can convert cylindrical to Cartesian coordinates using

$$\vec{e}_r = \cos\phi \vec{e}_x + \sin\phi \vec{e}_y \ , \qquad \vec{e}_\phi = -\sin\phi \vec{e}_x + \cos\phi \vec{e}_y \ . \tag{9}$$

The angle ϕ is computed anti-clockwise in the xy plane. In our case, the coil is in the zx plane, meaning $\phi = 0$. therefore the direction $\vec{e}_\phi$ in the plane $\phi = 0$ converts to $\vec{e}_y$. The ingoing flux $\mathrm{d}\vec{S} = \mathrm{d}x \vec{e}_y$ is

$$\Phi = \oint \vec{B} \cdot \mathrm{d}\vec{S} = \int_b^{a+b} \frac{\mu_0 I a}{2\pi} \frac{\mathrm{d}x}{x} = \frac{\mu_0 I a}{2\pi} \log \frac{b+a}{b} \tag{10}$$

The Faraday's law gives the induced electromotive force induced on the coil:

$$\Delta V = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -\frac{\mathrm{d}\Phi}{\mathrm{d}b} \frac{\mathrm{d}b}{\mathrm{d}t} = -\frac{\mathrm{d}\Phi}{\mathrm{d}b} v \tag{11}$$

where we used the fact that the only quantity depending on time is b and $v = \frac{\mathrm{d}b}{\mathrm{d}t}$. To explicit the calculation

$$\Delta V = -\frac{\mu_0 I a v}{2\pi} \frac{\mathrm{d}}{\mathrm{d}b} \log \frac{b+a}{b} = -\frac{\mu_0 I a v}{2\pi} \left(\frac{1}{b+a} - \frac{1}{b} \right) = -\frac{\mu_0 I a v}{2\pi} \frac{a}{b(b+a)} \tag{12}$$

The square loop has a resistance R. Calculate the induced current $i(t)$ in the square loop.

The induced current is

$$i = \frac{\Delta V}{R} = -\frac{\mu_0 I v}{2\pi R} \frac{a^2}{b(b+a)} \tag{13}$$

The current flows in the anti-clockwise direction.

Calculate the power dissipated due to the Joule effect.

The power dissipated by the Joule effect is

$$P = i^2 R = \left(\frac{\mu_0 I v}{2\pi R}\frac{a^2}{b(b+a)}\right)^2 R \tag{14}$$

Calculate the Laplace force on the square loop.

The Laplace force is given by the contribution of the two pieces of wires parallel to the infinite wire

$$\vec{F}_L = \frac{\mu_0 I i a}{2\pi}\left(-\frac{1}{b}+\frac{1}{a+b}\right)\vec{e}_x \tag{15}$$

Repeat the previous questions, considering the distance b fixed and the current in the wire of the form $I(t) = I_0\cos\omega t$.

The flux of the magnetic field through the coil is the same as before

$$\Phi = \frac{\mu_0 I a}{2\pi}\log\frac{b+a}{b}\ . \tag{16}$$

The Faraday's law gives the induced electromotive force induced on the coil:

$$\Delta V = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -\frac{\mu_0 a}{2\pi}\log\frac{b+a}{b}\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\mu_0 a\omega I_0\sin\omega t}{2\pi}\log\frac{b+a}{b} \tag{17}$$

The induced current is

$$i = \frac{\Delta V}{R} = \frac{\mu_0 a\omega I_0\sin\omega t}{2\pi}\log\frac{b+a}{bR} \tag{18}$$

The power dissipated by the Joule effect is

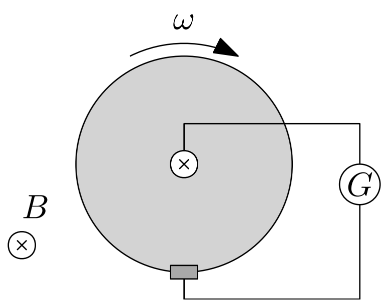
$$P = i^2 R = \left(\frac{\mu_0 a\omega I_0\sin\omega t}{2\pi}\log\frac{b+a}{bR}\right)^2 R \tag{19}$$

The Laplace force is again

$$\vec{F}_L = \frac{\mu_0 I i a}{2\pi}\left(-\frac{1}{b}+\frac{1}{a+b}\right)\vec{e}_x\ . \tag{20}$$

Exercise 4 - Faraday's Disk

Among the many experiments conducted by Faraday to study the phenomenon of induction, one was dedicated to demonstrating that a current appears in a moving conductor in a magnetic field. For this purpose, he considered a conductive disk able to rotate around its axis and be placed in a uniform magnetic field collinear with the disk's axis. A circuit containing a galvanometer connected the center of the disk to the edge of the disk via a sliding contact (see figure). Faraday observed that when the disk rotated, the needle of the galvanometer experienced a deflection.



Consider a disk with axis $\vec{e}_z$, radius R, and thickness a, rotating at speed ω and placed in a uniform magnetic field $\vec{B} = B\vec{e}_z$.

Explain the origin of the induced current. Calculate the electromotive force. Numerical application: $B = 0.2T$, $R = 0.1m$, $\omega = 50\text{rad}/s$.

The induced current in the rotating disk originates from the interaction between the magnetic field and the moving charges within the conductor (the disk, in this case). According to Faraday's law of electromagnetic induction, a changing magnetic flux through a conductor induces an electromotive force, and consequently, if there is a closed path, a current is induced. The electromotive force can be calculated by considering the movement of a differential ring element within the disk, at a distance r from the center, with a thickness dr. The velocity $\vec{v}$ of this element due to the disk's rotation is $\vec{v} = \vec{\omega} \times \vec{r} = \omega r \vec{e}_z \times \vec{e}_r = \omega r \vec{e}_\phi$, where $\vec{\omega} = \omega \vec{e}_z$ is the angular velocity.

The differential EMF ($\mathrm{d}V$) induced in this ring due to its motion in the magnetic field is given by the Lorentz force acting on the charges, which is

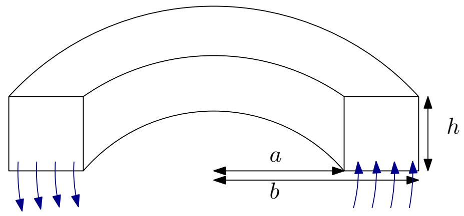
$$\mathrm{d}V = (\vec{v} \times \vec{B}) \cdot (\mathrm{d}r \vec{e}_r) = \omega r B \mathrm{d}r (\vec{e}_\phi \times \vec{e}_z) \cdot \vec{e}_r = \omega B r \mathrm{d}r \tag{21}$$

Integrating over all the points of the disk radius

$$\Delta V = \omega B \int_0^R r\mathrm{d}r = \frac{1}{2}\omega B R^2 = 0.5 \times 50Hz \times 0.2T \times 10^{-2}m^2 = 0.05V \tag{22}$$

Exercise 5 - Self-Inductance of a Toroidal Coil

Consider a toroidal coil with a square cross-section traversed by a current I (side h, inner radius a, outer radius b, N turns). Using Ampere's theorem, calculate the magnetic field and its flux.



Symmetry arguments tell us $\vec{B} = B(z,r)\vec{e}_\theta$. Using the Ampere's law on a circle inside the toroidal coil, we find

$$\int \vec{B} \cdot \mathrm{d}\vec{\ell} = \mu_0 I_{int} = \mu_0 N I \tag{23}$$

Using $\vec{B} \cdot \mathrm{d}\vec{\ell} = B(z,r) r \mathrm{d}\phi$ we find

$$B(z,r)2\pi r = \mu_0 N I 2\pi a \qquad \rightarrow \qquad \vec{B}(r) = \frac{\mu_0 N I}{2\pi r}\vec{e}_\phi \tag{24}$$

The flux through a square coil is

$$\Phi = \int \vec{B}(r) \cdot \mathrm{d}\vec{S} = \frac{\mu_0 N I}{2\pi} \int_a^b \frac{1}{r} h\mathrm{d}x = \frac{\mu_0 N I h}{2\pi} \log \frac{b}{a} \tag{25}$$

The flux through the entire toroidal coil is

$$\Phi_{tot} = N\Phi = \frac{\mu_0 N^2 I h}{2\pi} \log \frac{b}{a} \tag{26}$$

From the expression of the magnetic flux, deduce the self-inductance of the toroid L.

The self-inductance is

$$L = \frac{\Phi_{tot}}{I} = \frac{\mu_0 N^2 h}{2\pi} \log \frac{b}{a} \tag{27}$$

Calculate the total magnetic energy stored in the toroidal coil.

The magnetic energy density is

$$\mathcal{E}_B = \frac{B^2}{2\mu_0} = \frac{\mu_0 N^2 I^2}{8\pi^2} \frac{1}{r^2} \tag{28}$$

The total energy is the integral of $\mathcal{E}_B$ in the tous volume

$$E_B = \int \mathcal{E}_B \mathrm{d}V = \frac{\mu_0 N^2 I^2}{8\pi^2} h 2\pi \int_a^b \frac{1}{r^2} r\mathrm{d}r = \frac{\mu_0 N^2 I^2 h}{4\pi} \log \frac{b}{a} \tag{29}$$