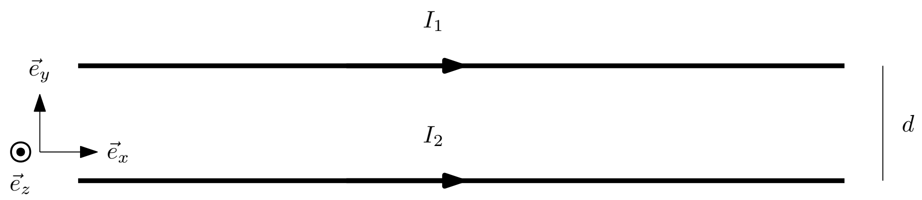
Exercise 1 - Laplace Force between Two Wires

Consider two infinite parallel conductive wires separated by a distance d. The wires carry a uniform current with intensity I_1 for the first wire and I_2 for the second wire. Calculate the Laplace force exerted by the first wire on the second. What is the force exerted by the second wire on the first?



The magnetic field B_1 created by the first wire at the location of the second wire is

$$B_1 = \frac{\mu_0 I_1}{2\pi d}(-\vec{e}_z) \tag{1}$$

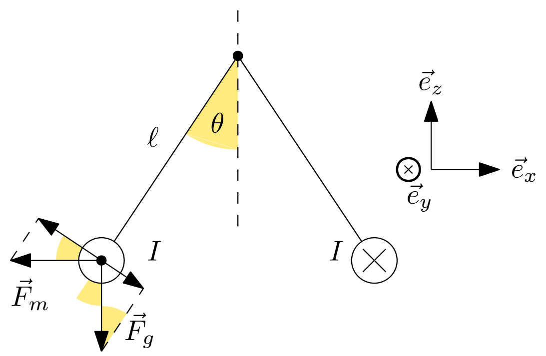
The force experienced by a current I_2 in a magnetic field B_1 per unit length is given by the Laplace force formula:

$$\vec{F} = \vec{I}_2 \times \vec{B}_1 = \frac{\mu_0 I_1}{2\pi d} I_2 \vec{e}_x \times (-\vec{e}_z) = \frac{\mu_0 I_1 I_2}{2\pi d}\vec{e}_y \tag{2}$$

Therefore, the two wires attract each other. The force exerted by the second wire on the first will be equal in magnitude and opposite in direction.

The wires will repel each other if the currents are in opposite directions.

Two long straight wires are suspended by strings of length $\ell = 4cm$ from a common axis. The wires have a mass density of $\lambda = 5 \times 10^{-2} kg/m$ and carry the same current but in opposite directions. What is this current if the strings make an angle of $\theta = 6$ degrees with the vertical?



The gravitational force by unit length of wire is

$$\vec{F}_g = -\lambda g \vec{e}_z \ , \tag{3}$$

where g is the gravitational constant. The previous exercise shows that the force between the two wires is repulsive and depends on their distance $d = 2\ell \sin\theta$.

$$\vec{F}_m = -\frac{\mu_0 I^2}{2\pi d}\vec{e}_x = -\frac{\mu_0 I^2}{4\pi \ell \sin\theta}\vec{e}_x \ . \tag{4}$$

The forces in the direction of the wire are balanced by the tension of the wire. The forces in the direction orthogonal to the wire are

$$F_m^\perp = \frac{\mu_0 I^2}{4\pi\ell\sin\theta}\cos\theta \quad \text{and} \quad F_g^\perp = -\lambda g \sin\theta \ . \tag{5}$$

If the system is in equilibrium

$$F_m^\perp + F_g^\perp = 0 \quad \to \quad \frac{\mu_0 I^2}{4\pi\ell\sin\theta}\cos\theta = \lambda g \sin\theta \ . \tag{6}$$

We can solve for I and find

$$I = \sqrt{\frac{4\pi\lambda g\ell}{\mu_0}\tan\theta\sin\theta}\ . \tag{7}$$

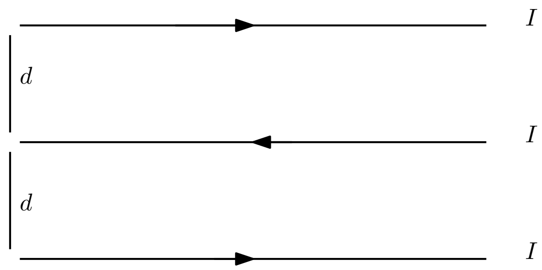
Substituting the values into the formula gives:

$$I = \sqrt{\frac{4\pi \times 5 \times 10^{-2}\,\mathrm{kg/m} \times 9.81\,\mathrm{m/s^2} \times 0.04\,\mathrm{m}}{4\pi \times 10^{-7}\,\mathrm{T\cdot m/A}}\tan(6^\circ)\sin(6^\circ)} \tag{8}$$

This calculation yields the current I of approximately $46.43A$.

Exercise 2 - Three infinite wires

Three infinite parallel conductive wires separated by a distance d are carrying a uniform current of intensity I in the direction indicated in the following figure. Calculate the magnetic force on each wire.



Using the result of exercise 1, we can compute the force per unit length on the first wire. There are two contributions. One is due to the magnetic field generated by the second wire, and the other is due to the magnetic field generated by the third wire.

$$\vec{F}_{\to 1} = \vec{F}_{2\to 1} + \vec{F}_{3\to 1} = \frac{\mu_0 I^2}{2\pi(d)}\vec{e}_z - \frac{\mu_0 I^2}{2\pi(2d)}\vec{e}_y = \frac{\mu_0 I^2}{4\pi d}\vec{e}_y \tag{9}$$

The resulting force pushes the first wire away from the other two. By symmetry arguments, the force on the last wire has to be the same but in the opposite direction

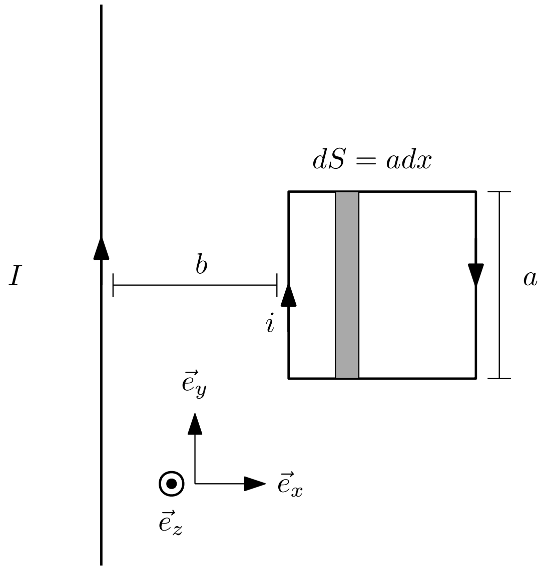
$$\vec{F}_{\to 3} = \vec{F}_{2\to 3} + \vec{F}_{1\to 3} = -\frac{\mu_0 I^2}{2\pi(d)}\vec{e}_z + \frac{\mu_0 I^2}{2\pi(2d)}\vec{e}_y = -\frac{\mu_0 I^2}{4\pi d}\vec{e}_y\ . \tag{10}$$

Finally, the force on the middle wire is

$$\vec{F}_{\to 2} = \vec{F}_{1\to 3} + \vec{F}_{2\to 3} = \frac{\mu_0 I^2}{2\pi(d)}\vec{e}_z - \frac{\mu_0 I^2}{2\pi(d)}\vec{e}_y = 0\ . \tag{11}$$

Exercise 3 - Flux of the Magnetostatic Field and Laplace Force

An infinitely long straight conductive wire carrying a current I is placed in the plane of a square loop of side a carrying a current i. The two conductors are placed at a distance b from each other (see drawing). Calculate the flux of $\vec{B}$ passing through the surface element dS centered on x (shaded area in the drawing) and deduce the flux of $\vec{B}$ through the complete circuit.



The magnetic field generated by the infinite wire is in direction $-\vec{e}_z$ and is given by

$$\vec{B} = -\frac{\mu_0 I}{2\pi x}\vec{e}_z \tag{12}$$

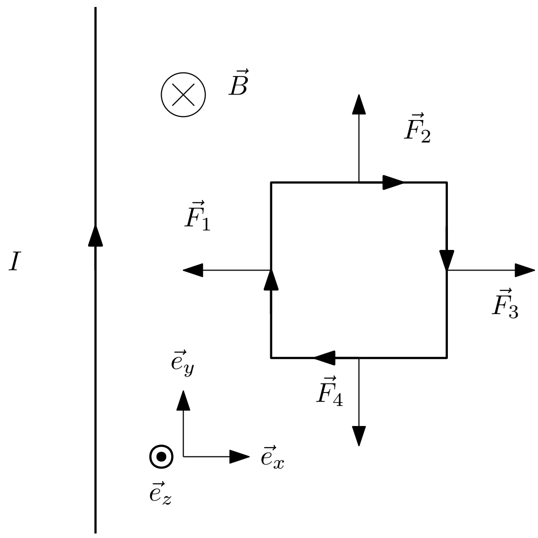
The flux on the shaded area $\mathrm{d}\vec{S} = \mathrm{d}x\vec{e}_z$ is

$$d\Phi = \vec{B}\cdot\mathrm{d}\vec{S} = -\frac{\mu_0 I}{2\pi x}\vec{e}_z\cdot\vec{e}_z a\mathrm{d}x = -\frac{\mu_0 I a}{2\pi}\frac{\mathrm{d}x}{x}\ . \tag{13}$$

The flux on the whole square is the integral

$$\Phi = \int_b^{b+a} d\Phi = -\frac{\mu_0 I a}{2\pi}\int_b^{b+a}\frac{\mathrm{d}x}{x} = -\frac{\mu_0 I a}{2\pi}\log\frac{b+a}{b}\ . \tag{14}$$

Provide the expression for the force exerted by the straight conductor on the square circuit. Deduce the motion of the circuit (attraction or repulsion).



Let's split the force into four contributions due to the interaction of the four pieces of the square wire with the magnetic field. We start with the contribution of the top piece and the bottom pieces. We can say that for symmetry reasons

$$\vec{F}_2 + \vec{F}_4 = 0\ . \tag{15}$$

However, we can also compute them explicitly and see it. Starting from $\vec{F}_2$, the force due to an infinitesimal piece of the square wire is the Lorentz force

$$\mathrm{d}\vec{F}_2 = q\vec{v} \times \vec{B} = i\mathrm{d}\vec{\ell} \times \vec{B} = i\mathrm{d}x\vec{e}_x \times \left(-\frac{\mu_0 I}{2\pi x}\vec{e}_z\right) = \frac{\mu_0 i I}{2\pi}\frac{\mathrm{d}x}{x}\vec{e}_y \ . \tag{16}$$

Therefore the total force is

$$\vec{F}_2 = \int_b^{b+a} \mathrm{d}\vec{F}_2 = \frac{\mu_0 i I}{2\pi}\int_b^{b+a}\frac{\mathrm{d}x}{x}\vec{e}_y = \frac{\mu_0 i I}{2\pi}\log\frac{b+a}{b}\vec{e}_y \ . \tag{17}$$

The calculation for F_4 is the same, but the sign of the current is opposite $-i$

$$\vec{F}_4 = -\frac{\mu_0 i I}{2\pi}\log\frac{b+a}{b}\vec{e}_y \ . \tag{18}$$

From which $\vec{F}_2 + \vec{F}_4 = 0$. The force $\vec{F}_1$ is

$$\vec{F}_1 = ia\vec{e}_y \times \vec{B} = ia\vec{e}_y \times \left(-\frac{\mu_0 I}{2\pi b}\vec{e}_z\right) = -\frac{\mu_0 I i}{2\pi}\frac{a}{b}\vec{e}_x \ . \tag{19}$$

The force $\vec{F}_2$ is

$$\vec{F}_2 = (-ia\vec{e}_y) \times \vec{B} = (-ia\vec{e}_y) \times \left(-\frac{\mu_0 I}{2\pi(a+b)}\vec{e}_z\right) = \frac{\mu_0 I i}{2\pi}\frac{a}{a+b}\vec{e}_x \ . \tag{20}$$

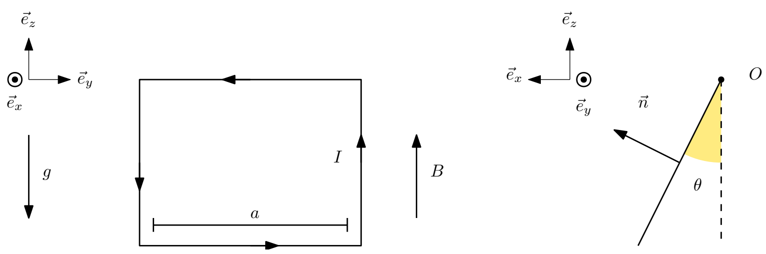
The total force is

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4 = \frac{\mu_0 I i}{2\pi}\left(\frac{a}{a+b} - \frac{a}{b}\right)\vec{e}_x = -\frac{\mu_0 I i}{2\pi}\frac{a^2}{b(a+b)}\vec{e}_x \ . \tag{21}$$

The force is attractive.

Exercise 4 - Rotation of a coil

Consider a coil formed by N identical square loops, each with a side length of a, suspended from a horizontal side (figure). Under the action of gravity $\vec{g}$ and a vertical magnetic field $\vec{B}$, the coil makes an angle θ with the vertical in its equilibrium position. Perform a numerical analysis with $a = 10cm$, $I = 0.1A$, $m = 80g$, $N = 200$, and $B = 115mT$. Calculate the dipole magnetic moment of this frame.



The dipole magnetic moment of the coil is given by

$$\vec{m} = I\vec{S} = INa^2\vec{n} \tag{22}$$

where we orient the surface as the current and $\vec{n} = \cos\theta\vec{e}_x + \sin\theta\vec{e}_z$ is the normal vector to the coil.

Calculate the moment of the Laplace force and the moment of the weight force at point O.

In a constant magnetic field, the four forces acting on the coil are

$$\vec{F}_1 = NIa(-\vec{e}_y) \times B\vec{e}_z = -aNIB\vec{e}_x \tag{23}$$

for the top section of the coil.

$$\vec{F}_2 = NIa(-\vec{e}_z\cos\theta + \vec{e}_x\sin\theta) \times B\vec{e}_z = -aNIB\sin\theta\vec{e}_y \tag{24}$$

for the left section of the coil.

$$\vec{F}_3 = NIa(\vec{e}_y) \times B\vec{e}_z = aNIB\vec{e}_x \tag{25}$$

for the bottom section of the coil.

$$\vec{F}_4 = NIa(\vec{e}_z \cos\theta - \vec{e}_x \sin\theta) \times B\vec{e}_z = aNIB\sin\theta\vec{e}_y \tag{26}$$

for the right section of the coil. The only force with non-zero momenta respect to the axis $\vec{e}_y$ is $\vec{F}_3$ and its momenta is

$$\vec{M}_B = a(-\vec{e}_z \cos\theta + \vec{e}_x \sin\theta) \times \vec{F}_3 = a^2NIB(-\vec{e}_z \cos\theta) \times \vec{e}_x = -a^2NIB\cos\theta\vec{e}_y \tag{27}$$

Alternatively, we could just say that the magnetic force moment of a dipole magnetic moment $\vec{m}$ is

$$\vec{M}_B = \vec{m} \times \vec{B} = a^2INB\vec{n} \times \vec{e}_z = a^2INB\cos\theta\vec{e}_x \times \vec{e}_z = -a^2INB\cos\theta\vec{e}_y \tag{28}$$

The magnetic moment of the weight force can be computed by assuming that the force is exerted in the center of mass and

$$\vec{M}_g = \frac{a}{2}(-\vec{e}_z \cos\theta + \vec{e}_x \sin\theta) \times (-mg\vec{e}_z) = mg\frac{a}{2}\sin\theta\vec{e}_y \tag{29}$$

Determine the equilibrium angle θ_0.

At equilibrium

$$\vec{M} = \vec{M}_B + \vec{M}_g = 0 = a\left(-aINB\cos\theta_0 + \frac{mg}{2}\sin\theta_0\right)\vec{e}_y \tag{30}$$

Therefore we can solve for θ

$$\tan\theta_0 = \frac{2aINB}{mg} \qquad \rightarrow \qquad \theta_0 = \arctan\frac{2aINB}{mg} \tag{31}$$