Exercise 1 - Lorentz Force: Cyclotron Frequency

Electrons with a velocity of $v = 106 m/s$ enter a region with a uniform magnetic field. They follow a circular path with a radius of $10cm$. What is the angular velocity of the electron? What is the magnetic field?

The initial transversal velocity of the electrons is $v = 106 m/s$. The angular velocity ω is related to the transversal velocity v by

$$\omega = \frac{v}{R} = \frac{106}{0.1} s^{-1} = 1060 Hz \tag{1}$$

The force on the electron is the Lorentz force

$$\boxed{\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) \ .} \tag{2}$$

There is no electric field. The motion is circular. The magnetic force cancels the centrifugal force

$$q\vec{v} \times \vec{B} = m\omega^2 R\vec{e}_r \ . \tag{3}$$

Since $v = \omega R\vec{e}_\phi$ we have

$$q\omega R\vec{e}_\phi \times \vec{B} = m\omega^2 R\vec{e}_r \ . \tag{4}$$

By simplifying and taking the cross-product with $\vec{e}_\phi$, we find

$$\vec{B} - (\vec{e}_\phi \cdot \vec{B})\vec{e}_\phi = \frac{m\omega}{q}\vec{e}_z \ . \tag{5}$$

Which is solved by

$$\vec{B} = \frac{m\omega}{q}\vec{e}_z = \frac{mv}{qR}\vec{e}_z \ . \tag{6}$$

We used the properties of the cross-product

$$\boxed{\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c} \ ,} \tag{7}$$

to prove that

$$\vec{e}_\phi \times (\vec{e}_\phi \times \vec{B}) = (\vec{e}_\phi \cdot \vec{B})\vec{e}_\phi - (\vec{e}_\phi \cdot \vec{e}_\phi)\vec{B} \ . \tag{8}$$

Exercise 2 - Proton in a magnetic field

A proton moves in a uniform magnetic field $B = 0.5T$ applied along the x axis. At $t = 0$, the proton has a velocity $v_x = 1.5 \times 10^5 m/s$, $v_y = 0$, $v_z = 2 \times 10^5 m/s$. Calculate the Lorentz force acting on the proton and its acceleration at $t = 0$. Determine the radius of helical motion and the pitch of the helix (the distance traveled along the axis of the helix in one period).

The Lorentz force on the proton at time $t = 0$ is

$$\vec{F} = q\vec{v} \times \vec{B} = qv_z B\,\vec{e}_z \times \vec{e}_x = qv_z B\,\vec{e}_y \ . \tag{9}$$

To solve the equations of motion is useful to use cylindrical coordinates. The general Lorentz force is radial and will balance the centrifugal force. There are no forces in the x direction. The motion of the particle is helical with a constant speed in the x direction equal to the initial one and a circular motion in the yz plane with angular velocity (see the previous exercise)

$$\omega = \frac{Bq}{m} \ . \tag{10}$$

The radius of the angular motion is computed in terms of the initial angular velocity

$$R = \frac{v}{\omega} = \frac{mv_z}{Bq} \ . \tag{11}$$

The period of one revolution is

$$T = \frac{2\pi}{\omega} = \frac{2\pi m}{qB} \ , \tag{12}$$

and the pitch is

$$p = v_x T = \frac{2\pi m v_x}{qB} \ . \tag{13}$$

Exercise 3 - Mass Spectrometer

A Selenium ion (Se^-) enters a region where there is an electric field $E = 1.12 \times 10^5 V/m$ and a uniform magnetic field $B = 0.54T$, both perpendicular to the ion's velocity and to each other. Calculate the particle's velocity if it passes through this region undeflected.

Suppose for simplicity that $\vec{E} = E\vec{e}_x$ and $\vec{B} = B\vec{e}_y$ while the velocity of the particle is along the z direction $\vec{v} = v\vec{e}_z$. The force has to vanish if the particle passes through this region undeflected. The Lorentz force is

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) = 0 \tag{14}$$

Plugging the electric and magnetic fields

$$0 = q\left(E\vec{e}_x + vB\,\vec{e}_z \times \vec{e}_y\right) = q\left(E - vB\right)\vec{e}_x \qquad \to \qquad v = \frac{E}{B}\ . \tag{15}$$

Doing the numerical analysis

$$v = \frac{1.12 \times 10^5 V/m}{0.54T} = 2.07 \times 10^5 m/s\ . \tag{16}$$

Upon exiting this region, the ion enters a zone where there is a uniform magnetic field $B = 0.54T$ perpendicular to its velocity. Knowing that the ion follows a circular trajectory with a radius $r = 31cm$, determine the mass of the Selenium ion and the mass number (A) of the isotope.

As we derived in the first exercise, the relation between velocity, radius, and magnetic field

$$m = -\frac{qRB}{v} = \frac{1.602 \times 10^{-19}C0.31m0.54T}{2.07 \times 10^5 m/s} = 1.29 \times 10^{-25}kg\ . \tag{17}$$

The atomic mass is the mass divided by the atomic mass unit

$$A = \frac{m}{1u} = \frac{1.29 \times 10^{-25}kg}{1.66 \times 10^{-27}kg} = 78\ , \tag{18}$$

which is compatible with the selenium mass number.

Exercise 4 - Lorentz Force: Isotope Separation

One method of separating the isotopes ^{235}U and ^{238}U of uranium (92 protons) was based on the difference in the radii of their trajectories in a magnetic field. Once ionized, the atoms are assumed to originate from a common source and move perpendicular to the field. Find the maximum spatial separation of the beams with a radius of $0.5\,m$ for ^{235}U in a field of $1.5\,T$ if the isotopes have the same velocity or if the isotopes have the same energy.

We start with the case where the two isotopes have the same velocity. For isotopes moving at the same velocity, the radius of their circular path in the magnetic field is given by

$$R = \frac{mv}{qB}\ , \tag{19}$$

where m is the mass, v is the velocity, q is the charge, and B is the magnetic field strength. The radius for the isotopes ^{238}U and ^{235}U, with masses m_{238} and m_{235}, respectively, are

$$R_{238} = \frac{m_{238}v}{qB} \qquad R_{235} = \frac{m_{235}v}{qB} \tag{20}$$

The ratio of the two radii is

$$\frac{R_{238}}{R_{235}} = \frac{m_{238}}{m_{235}}\ , \qquad \to \qquad R_{238} = \frac{m_{238}}{m_{235}}R_{235} = \frac{238}{235}0.5m = 0.506m\ , \tag{21}$$

which corresponds to a $6mm$ displacement. If the two isotopes have the same kinetic energy, we have to express their velocity in terms of the kinetic energy

$$K = \frac{1}{2}mv^2\ , \qquad \to \qquad v = \sqrt{\frac{2K}{m}}\ . \tag{22}$$

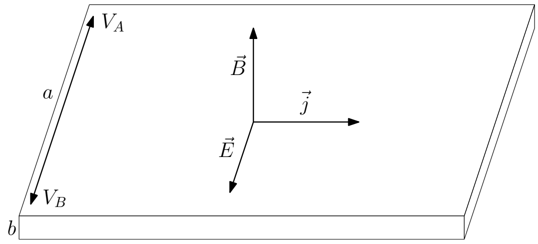
The ratio of the radii is

$$\frac{R_{238}}{R_{235}} = \frac{m_{238}v_{238}}{m_{235}v_{235}} = \sqrt{\frac{m_{238}}{m_{235}}} \ , \qquad \rightarrow \qquad R_{238} = \sqrt{\frac{m_{238}}{m_{235}}} R_{235} = \sqrt{\frac{238}{235}} 0.5m = 0.503m \ , \tag{23}$$

which corresponds to a $3mm$ displacement.

Exercise 5 - The Hall Effect

The Hall effect, discovered in 1880, is related to the appearance of a potential difference (i.e., an electric field) when a magnetic field $\vec{B}$ is applied perpendicularly to a conductor carrying a current $\vec{j}$.



Calculate the Hall field from the equilibrium condition of a conductor's electron.

The electrons in the conductor are subject to the Lorentz force. The equilibrium condition requires $\vec{F} = 0$.

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) = 0 \tag{24}$$

That means $E = vB$ assuming that the trios of E, B, and j are orthogonal.

Determine the Hall constant R_H as a function of the number of carriers (n is the carrier density); by definition, we write $\vec{E} = R_H \vec{B} \times \vec{j}$. The relationship between the carrier density n and the current density is $\vec{j} = nq\vec{v}$.

Since

$$E = vB = \frac{1}{nq} jB \ , \qquad \rightarrow \qquad R_H = \frac{1}{nq} \ . \tag{25}$$

On an indium arsenide InAs probe, a magnetic field of $B = 37mT$ is applied, and a current $I = 150mA$ is measured for a potential difference of $V = 4.7mV$ (thickness $b = 0.12mm$), calculate the Hall constant and the number of carriers per unit volume.

The Hall electric field is uniform. Therefore we compute it directly from the potential difference $E = V/a$. The current density is the total current divided by the sectional area of the conductor $j = I/(ab)$. The Hall constant

$$R_H = \frac{E}{jB} = \frac{V}{a}\frac{ab}{IB} = \frac{Vb}{IB} \ . \tag{26}$$

Or numerically

$$R_H = \frac{Vb}{IB} = \frac{4.7 \times 10^{-3}V \times 1.2 \times 10^{-4}m}{150 \times 10^{-3}A \times 37 \times 10^{-3}T} = 3.67 \times 10^{-5}\,\frac{m^3}{C} \ . \tag{27}$$

The carriers are electrons, and the electrons' density is

$$n = \frac{1}{R_H e} \approx 1.7 \times 10^{23} m^{-3} \ . \tag{28}$$