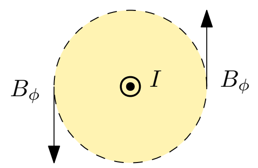
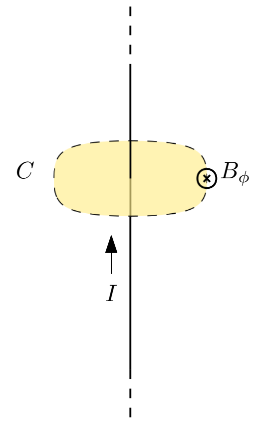
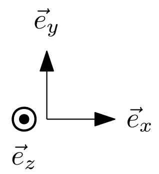
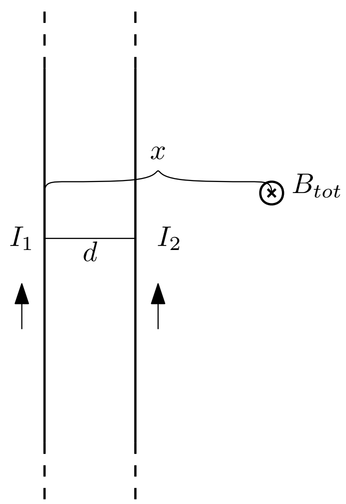
Exercise 1 - Magnetic field generated by parallel wires

Two infinite straight parallel wires are located at a distance of $d = 40cm$ from each other. One wire carries a current of $I_1 = 25A$, and the second carries a current of $I_2 = 3I_1 = 75A$. Find the points where the magnetic field created by the two wires cancels out. Consider both cases: the currents flow in the same direction and opposite directions.









First, we compute the magnetic field of one wire. Using the right-hand-rule and the symmetry of the system, we find that

$$\vec{B} = B_\phi \vec{e}_\phi \ . \tag{1}$$

We apply Ampere's circuital law on a circle centered on the wire

$$\boxed{\oint_C \vec{B} \cdot \mathrm{d}\vec{\ell} = \mu_0 I_{int}} \tag{2}$$

where μ_0 is the magnetic constant and I_{int} is the current crossing the surface delimited by the circle C. Since $\vec{B} = B_\phi \vec{e}_\phi$ with B_ϕ constant on the circle, we can integrate Ampere's law directly

$$\oint_C \vec{B} \cdot \mathrm{d}\vec{\ell} = B_\phi\, 2\pi r = \mu_0 I_{int} \ , \qquad \rightarrow \qquad B_\phi = \frac{\mu_0 I_{int}}{2\pi r} \ . \tag{3}$$

The magnetic field is linear. The magnetic field of the two wires is the sum of the magnetic fields of the wires alone.

$$\vec{B}_{tot} = \vec{B}_1 + \vec{B}_2 \tag{4}$$

To find the points in space with vanishing magnetic fields $\vec{B}_{tot} = 0$, we must look for the points where the two magnetic fields have the same direction $\vec{B}_1 = -\vec{B}_2$. The magnetic fields are both radial in their own coordinate system. The two radial directions coincide only when they intersect the plane containing the two wires. We use Cartesian coordinates to describe the magnetic field on the plane, and we assume that the currents flow in the same direction.

$$\vec{B}_{tot} = \frac{\mu_0 I_1}{2\pi x}(-\vec{e}_z) + \frac{\mu_0 I_2}{2\pi(x-d)}(-\vec{e}_z) = -\frac{\mu_0}{2\pi}\vec{e}_z \left(\frac{I_1}{x} + \frac{I_2}{x-d}\right) \ . \tag{5}$$

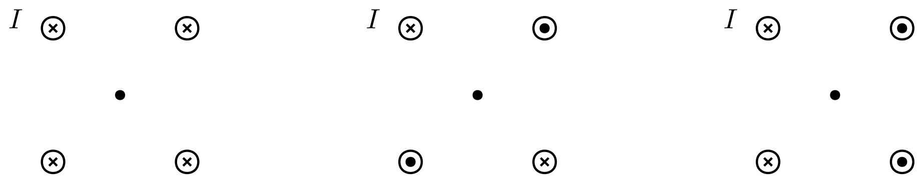
If $I_2 = 3I_1$ and we require $\vec{B}_{tot} = 0$ we find

$$\frac{1}{x} = \frac{3}{d-x} \ , \qquad \rightarrow \qquad 3x = d - x \ , \qquad \rightarrow \qquad x = \frac{d}{4} \ . \tag{6}$$

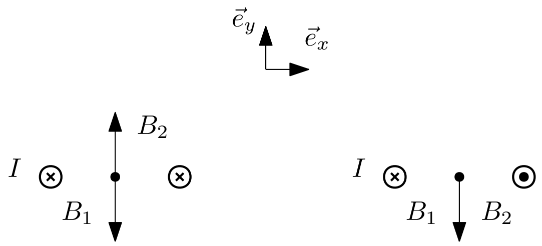
If the currents flow in opposite directions $I_2 \rightarrow -I_2$ and

$$\vec{B}_{tot} = -\frac{\mu_0}{2\pi}\vec{e}_z \left(\frac{I_1}{x} - \frac{I_2}{x-d}\right) = 0 \ , \qquad \rightarrow \qquad x = -\frac{d}{2} \ . \tag{7}$$

Four infinite straight parallel wires are each carrying a current of $I = 100A$. The four wires are located at the vertices of a square with a side length of $\ell = 10cm$ (above). Calculate the magnetic field at the center of the square for the following 3 cases:



First, we look at the simpler case of two parallel wires at a distance d carrying the same current I in the same direction.



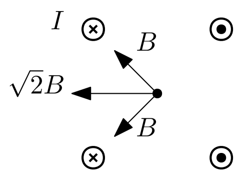
We compute the magnetic field at a distance $d/2$ using the result of the previous point

$$\vec{B}_{tot} = -\frac{\mu_0 I}{2\pi x}\vec{e}_y - \frac{\mu_0 I}{2\pi(x-d)}\vec{e}_y \qquad \text{for } x = \frac{d}{2} \rightarrow \qquad \vec{B}_{tot} = -\frac{\mu_0 I}{\pi d}\vec{e}_y + \frac{\mu_0 I}{\pi d}\vec{e}_y = 0 \ . \tag{8}$$

If the currents flow in opposite directions

$$\vec{B}_{tot} = -\frac{\mu_0 I}{2\pi x}\vec{e}_y + \frac{\mu_0 I}{2\pi(x-d)}\vec{e}_y \qquad \text{for } x = \frac{d}{2} \rightarrow \qquad \vec{B}_{tot} = -\frac{\mu_0 I}{\pi d}\vec{e}_y - \frac{\mu_0 I}{\pi d}\vec{e}_y = 2\frac{\mu_0 I}{\pi d}(-\vec{e}_y) \ . \tag{9}$$

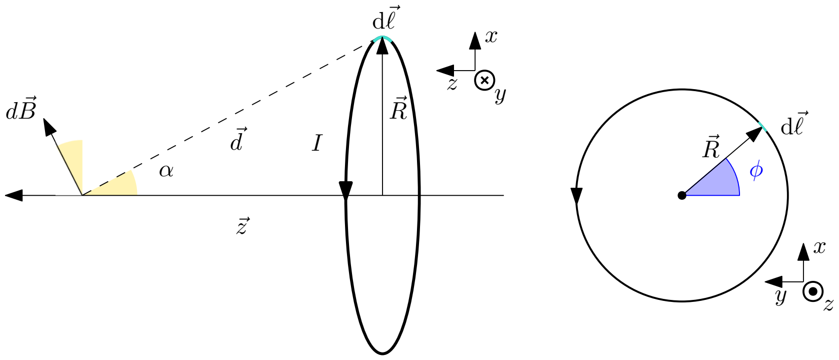
To solve the exercise, we first consider the two pairs of wires at opposite corners of the squares. For the first two cases, the currents of opposite cables flow in the same direction. The analysis we did with two wires shows that the magnetic field is vanishing. In the last case, the magnetic fields sum, the distance between opposite vertices of the square is the diagonal $d = \sqrt{2}\ell$,



$$\vec{B}_{tot} = 2\sqrt{2}\frac{\mu_0 I}{\pi d}(-\vec{e}_x) = 2\frac{\mu_0 I}{\pi \ell}(-\vec{e}_x) \ . \tag{10}$$

Exercise 2 - Current-carrying Loop

Consider a circular loop of radius R carrying a current of intensity I. Determine the magnetic field vector $\vec{B}$ at a point M on its axis Oz, first as a function of the angle α under which the loop is seen from point M, and then as a function of the z coordinate of point M. What is the value of the field at the center of the loop? Plot the variations of $\vec{B}$ as a function of z.



The current in an infinitesimal piece of the loop $\mathrm{d}\vec{\ell}$ generates a magnetic field (law of Biot-Savart)

$$\mathrm{d}\vec{B} = \frac{\mu_0 I}{4\pi}\frac{\mathrm{d}\vec{\ell}\times\vec{d}}{d^3} \; . \tag{11}$$

We assume that the loop lies in the xy plane and the axis of the loop is in the z direction. The vector

$$\vec{d} = \vec{z} - \vec{R} \quad \text{with} \quad \vec{R} = R\cos\phi\,\vec{e}_x + R\sin\phi\,\vec{e}_y \;, \quad \text{and} \quad \mathrm{d}\vec{\ell} = R\vec{e}_\phi = -R\sin\phi\,\vec{e}_x + R\cos\phi\,\vec{e}_y \; . \tag{12}$$

We can argue for symmetry reasons that the magnetic field must be in the z direction. We can also brute force the calculation. The cross product

$$\mathrm{d}\vec{\ell}\times\vec{d} = zR\cos\phi\,\vec{e}_x + zR\sin\phi\,\vec{e}_y + R^2\,\vec{e}_z \tag{13}$$

and $d = \sqrt{R^2+z^2}$, $\cos\alpha = \frac{z}{\sqrt{R^2+z^2}}$, and $\sin\alpha = \frac{R}{\sqrt{R^2+z^2}}$. Finally,

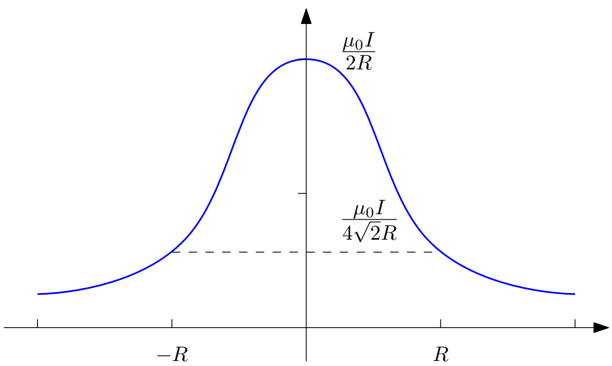
$$\vec{B} = \frac{\mu_0 I R}{4\pi}\int_0^{2\pi}\frac{\sin^3\alpha}{R^3}\left(z\cos\phi\,\vec{e}_x + z\sin\phi\,\vec{e}_y + R\,\vec{e}_z\right)\mathrm{d}\phi = \tag{14}$$

$$\frac{\mu_0 I}{2}\frac{\sin^3\alpha}{R}\vec{e}_z = \frac{\mu_0 I}{2}\frac{R^2}{(\sqrt{R^2+z^2})^3}\vec{e}_z \; . \tag{15}$$

At the center of the loop $z = 0$ and

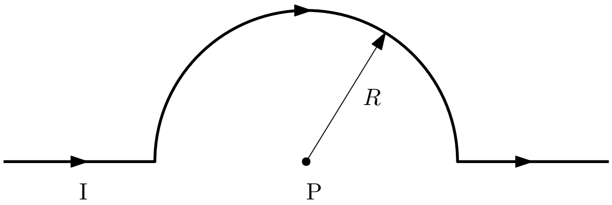
$$\vec{B} = \frac{\mu_0 I}{2R}\vec{e}_z \; . \tag{16}$$

The plot of the magnetic field on the loop axis is

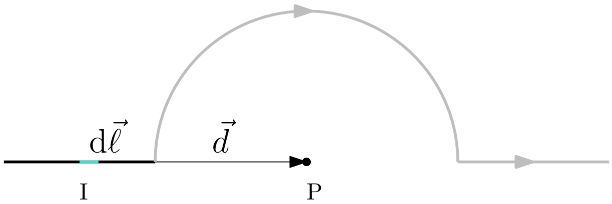


Exercise 3 - Half loop

Determine the magnetic field at point P created by the current flowing through the circular part of the wire.



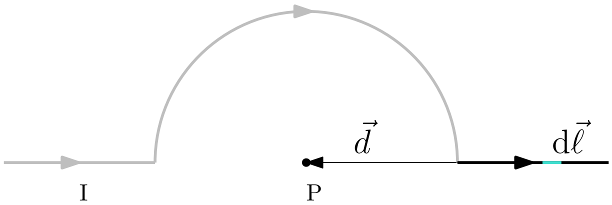
We compute the magnetic field at the point P using the Biot-Savart law. We divide the wire into three sections. The total magnetic field is the sum of the three contributions. The first section is the straight part of the wire on the left.



In this part of the wire the vector product $\mathrm{d}\vec{\ell} \times \vec{d} = 0$ since $\mathrm{d}\vec{\ell}$ is parallel to $\vec{d}$

$$\vec{B} = \int \frac{\mu_0 I}{4\pi} \frac{\mathrm{d}\vec{\ell} \times \vec{d}}{d^3} = 0 \ . \tag{17}$$

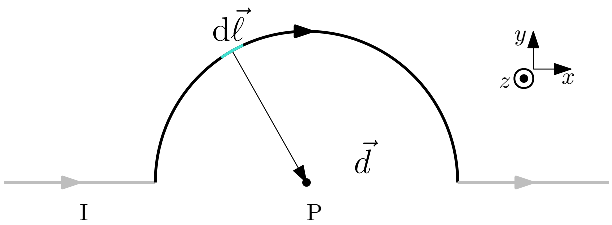
. The third section is the straight part of the wire on the right.



Also in this section the vector product $\mathrm{d}\vec{\ell} \times \vec{d} = 0$ since $\mathrm{d}\vec{\ell}$ is parallel to $\vec{d}$

$$\vec{B} = \int \frac{\mu_0 I}{4\pi} \frac{\mathrm{d}\vec{\ell} \times \vec{d}}{d^3} = 0 \ . \tag{18}$$

. The central section is the half-circular part of the wire

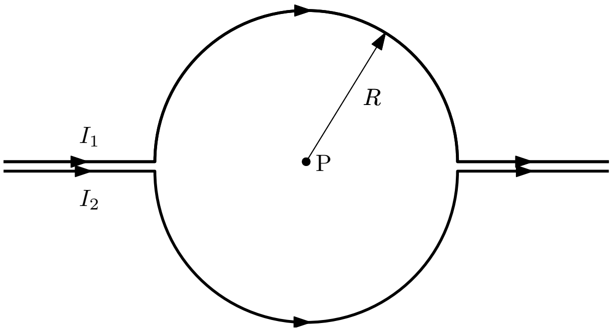


We have $\mathrm{d}\vec{\ell} = R\mathrm{d}\phi(-\vec{e}_\phi)$ and $\vec{d} = R(-\vec{e}_r)$, and $\mathrm{d}\vec{\ell} \times \vec{d} = -R^2\vec{e}_z\mathrm{d}\phi$. The magnetic field is

$$\vec{B} = \int \frac{\mu_0 I}{4\pi} \frac{\mathrm{d}\vec{\ell} \times \vec{d}}{d^3} = \int_0^\pi \frac{\mu_0 I}{4\pi} \mathrm{d}\phi \frac{R^2}{R^3} \vec{e}_z = -\frac{\mu_0 I}{4R} \vec{e}_z \ . \tag{19}$$

The minus sign indicates that the field is entering the half-loop plane. Notice that the magnitude is half the magnitude of the full loop.

Determine the magnetic field at point P created by the currents I_1 and I_2 flowing through the circular part of the two wires.



The magnetic field is the sum of the magnetic fields of the two half-loops. We computed the magnetic field of the top half in the previous point. The magnetic field of the lower half is the same with $I \to -I_2$ since the current is circulating in the anti-clockwise direction.

$$\vec{B} = -\frac{\mu_0 I_1}{4R}\vec{e}_z + \frac{\mu_0 I_2}{4R}\vec{e}_z = \frac{\mu_0 (I_2 - I_1)}{4R}\vec{e}_z \ . \tag{20}$$

Exercise 4 - Cylindrical conductors

An infinite solid cylindrical conductor of radius a carries a current of intensity I. Establish the expression for the magnetic field $vecB$ created by this conductor at a distance ρ in the following two cases: $\rho > a$ and $\rho < a$. Plot the variations of $|\vec{B}|$ as a function of ρ.

For reasons of symmetry, the magnetic field is radial. We use cylindrical coordinates and apply Ampere's law on circles of radius ρ concentric with the conductor. If $\rho > a$ then

$$\oint_{C_\rho} \vec{B} \cdot d\vec{\ell} = B\, 2\pi\rho = \mu_0 I \ , \qquad \to \qquad B = \frac{\mu_0 I}{2\pi\rho} \ . \tag{21}$$

If $\rho < a$, not all the current I is contained in the circle. The current in the circle can be calculated in terms of the current density $j = I/(\pi a^2)$

$$I_C = \pi\rho^2 j = \pi\rho^2 \frac{I}{\pi a^2} = I\frac{\rho^2}{a^2} \ . \tag{22}$$

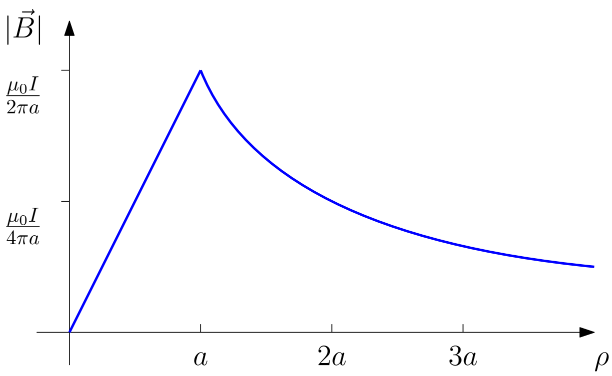
The Ampere's law for $\rho < a$ is

$$\oint_{C_\rho} \vec{B} \cdot d\vec{\ell} = B\, 2\pi\rho = \mu_0 I_C = \mu_0 I \frac{\rho^2}{a^2} \ , \qquad \to \qquad B = \frac{\mu_0 I}{2\pi a^2}\rho \ . \tag{23}$$

To summarize

$$\vec{B} = \begin{cases} \dfrac{\mu_0 I}{2\pi}\dfrac{1}{\rho}\,\vec{e}_\phi & \rho > a \\ \dfrac{\mu_0 I}{2\pi}\dfrac{\rho}{a^2}\,\vec{e}_\phi & \rho \le a \end{cases} \tag{24}$$

The magnetic field as a function of ρ is linear until $\rho = a$ and then decays as $1/\rho$.



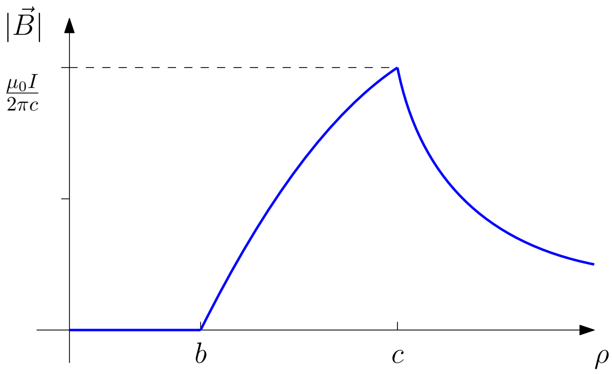
The same current I flows through a hollow cylindrical conductor with inner radius b and outer radius c. Establish the expression for the magnetic field $\vec{B}$ created by this conductor in the following three cases: $\rho < b$, $b < \rho < c$, and $\rho > c$. Plot the variations of $|\vec{B}|$ as a function of ρ.

The solution is similar to the previous case. Note that the magnetic field for $\rho < b$ is vanishing since there is no current. In the intermediate region $b \leq \rho \leq c$, the current inside a circle of radius ρ is

$$I_C = \pi(\rho^2 - b^2)\frac{I}{\pi(c^2 - b^2)} = I\frac{\rho^2 - b^2}{(c^2 - b^2)} \ . \tag{25}$$

The magnetic field outside the conductor is the same as the previous point.

$$\vec{B} = \begin{cases} 0 & \rho < b \\ \dfrac{\mu_0 I}{2\pi\rho}\dfrac{\rho^2 - b^2}{(c^2 - b^2)}\vec{e}_\phi & b \leq \rho \leq c \\ \dfrac{\mu_0 I}{2\pi}\dfrac{1}{\rho}\vec{e}_\phi & \rho > c \end{cases} \tag{26}$$



A coaxial cable is made of two long concentric conductive cylinders. The outer, hollow cylinder has an inner radius b and an outer radius c. The inner solid cylinder has a radius $a < b$. Both conductors carry equal (uniform) currents I but in opposite directions. Calculate the magnetic field of this system.

We assume that I is the current of the internal conductor and $-I$ is the current of the external conductor. The magnetic field in the first region $\rho \leq a$ is computed as in the first point of this exercise using Ampere's law

$$\vec{B} = \frac{\mu_0 I}{2\pi a^2}\rho\vec{e}_\phi \ . \tag{27}$$

In the second region between the Inner and Outer Cylinders $a < \rho < b$, the magnetic field is due to the current in the inner cylinder alone

$$\vec{B} = \frac{\mu_0 I}{2\pi\rho}\vec{e}_\phi \ . \tag{28}$$

In the region inside the Outer Cylinder $b < \rho < c$, the net enclosed current is the difference between the total current and the portion of the current in the outer cylinder corresponding to the area enclosed by the loop:

$$I_{\text{enc}} = I - I \cdot \frac{\rho^2 - b^2}{c^2 - b^2} = I\frac{c^2 - \rho^2}{c^2 - b^2} \ . \tag{29}$$

The magnetic field is then given by:

$$\vec{B} = \frac{\mu_0 I}{2\pi\rho}\frac{c^2 - \rho^2}{c^2 - b^2}\vec{e}_\phi \ . \tag{30}$$

Finally, in the region outside the Outer Cylinder $\rho > c$, the magnetic fields from the inner and outer cylinders cancel each other out, so the magnetic field is zero:

$$\vec{B} = 0 \ . \tag{31}$$

Exercise 5 - Solenoid

A solenoid is a circuit made of contiguous (but insulated) turns of a very thin conductor wound around a cylinder. Let L be the length of the cylinder, R the radius of its circular section, and N the total number of turns traversed by a constant current I. The axis of the solenoid is oriented in the direction of the current (right-hand rule). We will adopt the approximation of an infinite solenoid (i.e., $R \ll L$). Explain why the field $\vec{B}$ is parallel to the axis of the solenoid everywhere (symmetry). Using Ampère's theorem, argue that the magnetic field inside the solenoid, $\vec{B}_{int}$, is constant.

To find information on the direction of magnetic fields, the simplest method is to look for planes of symmetry. These are planes that leave the current distribution invariant by reflection. Since the magnetic field is a pseudo vector, a symmetry plane for the current is a plane of antisymmetry of the field. In this case, any plane parallel to the xy plane is a plane of symmetry for the current, so the magnetic field is necessarily along the direction z.

$$\vec{B}_{int} = B_{int}\vec{e}_z \tag{32}$$

The invariance under translation along the z axis and invariance under rotations around the z axis implies that B_{int} does not depend on z or ϕ. Therefore B_{int} depends only on ρ. We prove that B_{int} is also independent of ρ using Ampere's law on a rectangular path aligned with the axis x and z internal to the solenoid. Since the path does not intersect any current

$$\oint_R \vec{B}_{int} \cdot \mathrm{d}\vec{\ell} = 0 \ . \tag{33}$$

Since $\vec{B}_{int} = B_{int}(\rho)\vec{e}_z$ the part of the contour integral along $\vec{e}_x$ vanish because $\mathrm{d}\vec{\ell}$ is orthogonal to $\vec{B}_{int}$.

$$\oint_R \vec{B}_{int} \cdot \mathrm{d}\vec{\ell} = B_{int}(x_1)d - B_{int}(x_2)d = 0 \ , \qquad \to \qquad B_{int}(x_1) = B_{int}(x_2) \ , \tag{34}$$

for all x_1 and x_2 inside the solenoid. Therefore, B_{int} is constant inside the solenoid.

Show that $\vec{B}_{ext}$ outside the solenoid is constant and argue that $\vec{B}_{ext} = 0$. Use Ampère's theorem to deduce the value of the magnetic field inside the solenoid $\vec{B}_{int}$.

The symmetry arguments apply also outside the solenoid. Therefore, also outside the solenoid

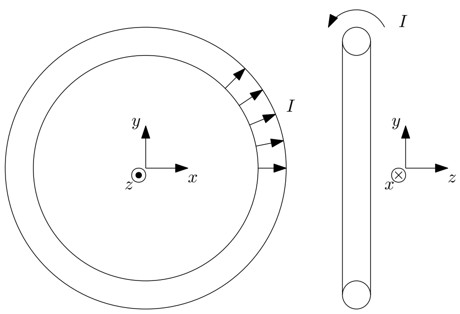
$$\vec{B}_{ext} = B_{ext}\vec{e}_z \tag{35}$$

with B_{ext} constant. The magnetic field in the plane xy infinitely distant from the solenoid has to be 0. Therefore, the magnetic field outside the solenoid is 0 everywhere. To compute the magnetic field inside the solenoid, we consider a rectangular path aligned with the axes x and z. We take one side outside the solenoid and one inside the solenoid. The current inside the path is dIN/L where d is the length of the rectangle in the z direction.

$$\oint_R \vec{B} \cdot \mathrm{d}\vec{\ell} = B_{int}(x_{int})d - B_{ext}(x_{ext})d = \mu_0 dI\frac{N}{L} \ , \qquad \to \qquad B_{int}(x_{int}) = \mu_0 I\frac{N}{L} \ . \tag{36}$$

Exercise 6 - Toroidal coil

A toroidal coil is made up of N turns regularly wound around a torus with a circular cross-section. We are interested in the value of the field in the xy plane. What is the value of the field at a point located at a distance r from the center of the coil?



The system has cylindrical symmetry. Any plane containing the z axis is a symmetry plane for the toroidal coil. The magnetic field has to be orthogonal to any of these planes. Therefore, it has an angular direction.

$$\vec{B} = B\vec{e}_\phi \tag{37}$$

Moreover, the system is invariant by rotations around the z axes, and B cannot depend on the angular variable ϕ. If we take a circle of radius r on the plane xy and apply the Ampere law on that circle and find.

$$\oint_C \vec{B}\cdot \mathrm{d}\vec{\ell} = B2\pi r = \mu_0 NI \ , \qquad -> \qquad B = \frac{\mu_0 NI}{2\pi r} \tag{38}$$

if r is inside the torus, 0 otherwise.

Show that assuming the radius of the coil is much larger than the radius of the torus's cross-section implies that the coil can be assimilated to an infinite solenoid.

If we write $r = R + \rho = R(1 + \frac{\rho}{R})$ with R the radius of the coil and $\rho = O(r_c)$ with r_c the radius of each loop. We expand in series $\frac{\rho}{R} \ll 1$ at the first order and find

$$B \approx \frac{\mu_0 NI}{2\pi R} \ , \tag{39}$$

which corresponds to the solenoid's magnetic field with $L = 2\pi R$.

Exercise 7 - Helmholtz's coils

Consider two circular coils of radius a, each consisting of N turns carrying a constant current I and located at a distance a from each other. The current flows in the same direction in both coils, producing a very uniform magnetic field in the region between them. Calculate the magnetic field along the z axis of the coils. Plot the shape of the $B(z)$ graph and compare it with the magnetic field produced by a single coil.

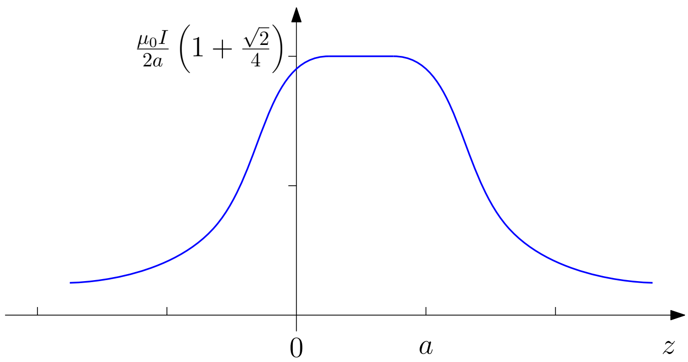
We computed the magnetic field of a loop on its axis in exercise 2.

$$\vec{B} = \frac{\mu_0 I}{2}\frac{a^2}{(\sqrt{a^2+z^2})^3}\vec{e}_z \ . \tag{40}$$

If we place one of them at $z = 0$ and the other one at $z = a$ and we compute the magnetic field along their axis by adding them

$$\vec{B} = \frac{\mu_0 I}{2}a^2\left(\frac{1}{(a^2+z^2)^{\frac{3}{2}}} + \frac{1}{(a^2+(a-z)^2)^{\frac{3}{2}}}\right)\vec{e}_z \ . \tag{41}$$

You see that there is an interval $(0, a)$ for which the magnetic field is almost constant.



Determine the magnetic field at point P (midpoint of the distance between the 2 coils). Calculate $\mathrm{d}B/\mathrm{d}z$ and $\mathrm{d}^2B/\mathrm{d}z^2$ at point P. Explain why the field is uniform near point P.

In the point $P = a/2$, the magnetic field is

$$\vec{B} = \frac{\mu_0 I}{2a}\frac{16\sqrt{5}}{25}\vec{e}_z \ . \tag{42}$$

After a long but simple calculation, we find that both

$$\left.\frac{\mathrm{d}\vec{B}}{\mathrm{d}z}\right|_P = 0 \qquad \left.\frac{\mathrm{d}^2\vec{B}}{\mathrm{d}z^2}\right|_P = 0 \qquad \left.\frac{\mathrm{d}^3\vec{B}}{\mathrm{d}z^3}\right|_P = 0. \tag{43}$$

The first non-zero derivative is the fourth derivative. This means that the magnetic field expanded around the midpoint P is constant up to fourth-order corrections.