Exercise 1 - Charge passing through a potential

What are the kinetic energy in Joules and the speed in m/s of an oxygen nucleus ^{16}O (8 protons, 8 neutrons) after acceleration by a potential difference of $\Delta V = 10^7 V$?

The energy of the system is conserved. The energy of the oxygen nucleus comprises the kinetic energy E_k and the electrostatic energy E_v

$$E = E_k + E_v \ . \tag{1}$$

The kinetic energy is given by

$$E_k = \frac{1}{2} m_o v^2 \ , \tag{2}$$

where $m_o = 8m_p + 8m_n$ is the mass of the oxygen nucleus ^{16}O in terms of the mass of the proton m_p and the mass of the neutron m_n. We will assume $m_p = m_n \approx 1.67 \times 10^{-27} kg$ therefore $m_o = 16 \cdot m_p = 26.72 \times 10^{-27} kg$. The electrostatic energy of a point particle with charge q in an electrostatic potential V is

$$\boxed{E_v = qV \ .} \tag{3}$$

The charge of the oxygen nucleus is equal to the charge of the eight protons $q_o = 8 \cdot e = 8 \cdot 1.60 \times 10^{-19} C = 12.8 \times 10^{-19} C$ The total energy is conserved. If the particle is initially at rest $v = 0$, its total energy is just electrostatic

$$E = q_o V_1 \ . \tag{4}$$

In the end, the energy is both kinetic and electrostatic

$$E = \frac{1}{2} m_o v^2 + q_o V_2 = q_o V_1 \ . \tag{5}$$

From this equation, we can determine E_k as a function of the potential difference $\Delta V = V_1 - V_2$

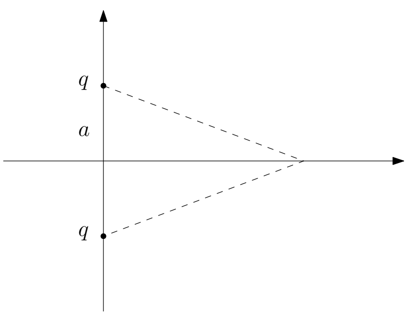
$$E_k = q_o \Delta V = 12.8 \times 10^{-19} C \cdot 10^7 V = 12.8 \times 10^{-12} J \ . \tag{6}$$

The final speed of the particle is

$$v = \sqrt{\frac{2E_k}{m_o}} = \sqrt{\frac{2 \cdot 12.8 \times 10^{-12} J}{26.72 \times 10^{-27} kg}} = 3.1 \times 10^7 m/s \ . \tag{7}$$

which is approximately 1/10 the **speed of light**.

Two identical positive charges ($q = 3nC$) are fixed on the y-axis at $y = a$ and $y = -a$ with $a = 10cm$. Calculate their potential on the x-axis and deduce the electrostatic field on the x-axis.



The electrostatic potential is additive. The distance of a point on the x axis from a charge is $d = \sqrt{x^2 + a^2}$. The full electrostatic potential is

$$V(x) = 2 \cdot \frac{q}{4\pi\epsilon_0} \frac{1}{\sqrt{x^2 + a^2}} = \frac{q}{2\pi\epsilon_0} \frac{1}{\sqrt{x^2 + a^2}} \ . \tag{8}$$

The electrostatic field on the x-axis is (minus) the gradient of the potential

$$\vec{E} = -\frac{\mathrm{d}}{\mathrm{d}x} V(x) \vec{e}_x = \frac{q}{2\pi\epsilon_0} \frac{x}{(a^2 + x^2)^{3/2}} \vec{e}_x \ . \tag{9}$$

A positive charge q' ($q' = 2nC$) with mass m ($m = 1g$) is now placed at rest near the origin, slightly offset towards positive x, then released. What will its speed be at infinity?

The initial total energy of the particle is just electrostatic

$$E = q' V(0^+) = q' \frac{q}{2\pi\epsilon_0} \frac{1}{a} \ . \tag{10}$$

The final total energy of the particle is just kinetic, since by convention $V(\infty) = 0$

$$E = \frac{1}{2} m v_f^2 \ . \tag{11}$$

The final velocity by conservation of energy

$$\frac{1}{2} m v_f^2 = q' \frac{q}{2\pi\epsilon_0} \frac{1}{a} \quad \rightarrow \quad v = \sqrt{\frac{q'q}{m\pi\epsilon_0} \frac{1}{a}} \ . \tag{12}$$

Numerically

$$v_f = \sqrt{4 \cdot 9 \times 10^9 N m^2 C^{-2} \cdot 3 \times 10^{-9} C \cdot 2 \times 10^{-9} \cdot 10^3 kg^{-1} \cdot 10 m^{-1}} \tag{13}$$

$$= \sqrt{4 \cdot 9 \cdot 3 \cdot 2 \times 10^{-5} m^2/s^2} = 0.046\, m/s \tag{14}$$

If the same charge is launched along the x-axis from $-\infty$ towards the origin with half the speed than that found in the previous point, at what distance from the origin will it turn back?

We use conservation of energy. The initial energy is only kinetical, and the final energy is only electrostatic. At the turning point, the particle has zero velocity

$$\frac{1}{2} m v_i^2 = \frac{q'q}{2\pi\epsilon_0} \frac{1}{\sqrt{x^2 + a^2}} \ . \tag{15}$$

We set $v_i = v_f/2$ the initial kinetic energy is 1/4 of the final kinetic energy of the previous question.

$$\frac{1}{4} \frac{q'q}{2\pi\epsilon_0} \frac{1}{a} = \frac{q'q}{2\pi\epsilon_0} \frac{1}{\sqrt{x^2 + a^2}} \ . \tag{16}$$

We can solve for x to find

$$16a^2 = x^2 + a^2 \quad x = -\sqrt{15} a \approx -38.7 cm \ . \tag{17}$$

The negative sign is because the particle is initially set at $x = -\infty$.

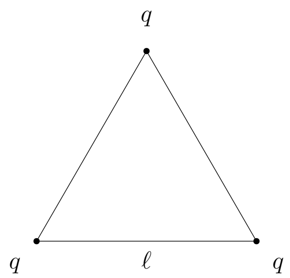
If a negative charge $-q'$ is released at rest on the x-axis from $-\infty$, what will its speed be when it passes the origin?

The initial energy is 0, and from the conservation of energy, we have

$$0 = \frac{1}{2} m v^2 + (-q') \frac{q}{2\pi\epsilon_0} \frac{1}{a} \quad \rightarrow \quad v = \sqrt{\frac{q'q}{m\pi\epsilon_0} \frac{1}{a}} \approx 0.046\, m/2 \tag{18}$$

Exercise 2 - Potential Energy

Three identical point charges with $q = 1.2\mu C$ are at the three vertices of an equilateral triangle with side $\ell = 0.5m$. Calculate the potential energy of the system.



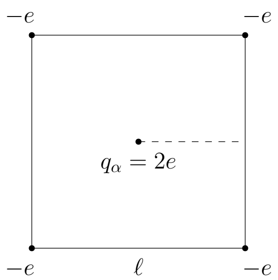
The potential energy of one particle due to the electrostatic potential of another particle is

$$V_{1-1} = \frac{1}{4\pi\epsilon_0}\frac{q^2}{\ell} . \quad (19)$$

The potential energy of each particle gets contributions from the electromagnetic potential of both other particles. To compute the total potential energy, we must sum up the potential energy of all three particles.

$$V = 3\frac{1}{4\pi\epsilon_0}\frac{q^2}{\ell} . \quad (20)$$

Four electrons are located at the vertices of a square with side $\ell = 10nm$, and an alpha particle (charge $q_\alpha = +2e$) is located at the center of the square. Calculate the work required to move the alpha particle from the square's center to the side's midpoint.



The initial energy of the α particle is just electrostatic and gets an equal contribution from all four electrons

$$E_i = 4(2q)\frac{1}{4\pi\epsilon_0}\frac{-q}{\ell\sqrt{2}/2} = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{\ell}8\sqrt{2} . \quad (21)$$

The final potential energy is

$$E_f = 2(2q)\frac{1}{4\pi\epsilon_0}\frac{-q}{\ell/2} + 2(2q)\frac{1}{4\pi\epsilon_0}\frac{-q}{\ell\sqrt{5}/2} = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{\ell}8(1+\frac{1}{\sqrt{5}}) . \quad (22)$$

The work in a conservative field is **minus** the variation of the potential energy

$$W = -E_f + E_i = \frac{1}{4\pi\epsilon_0}\frac{q^2}{\ell}8(1+\frac{1}{\sqrt{5}} - \sqrt{2}) \approx 6.08 \times 10^{-21}J . \quad (23)$$

Exercise 3 - Energy of a dipole

A dipole can be modeled by two point charges q and $-q$ separated by a distance $\vec{a}$. The quantity $\vec{p} = q\vec{a}$ is called the dipole moment. Determine the intrinsic energy U_0 of the dipole assuming that, with the position of the charge $+q$ fixed, we bring the charge $-q$ from infinity to a distance a.

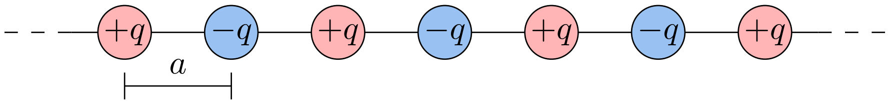
The intrinsic energy U_0 of the dipole can be calculated by considering the work done to assemble the dipole, which is the work done to bring the $-q$ charge from infinity to its position in the dipole. The potential energy at infinity distance is 0. This work done is equivalent to the potential energy of the $-q$ charge in the electric field created by the $+q$ charge, given by:

$$U_0 = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{a} . \quad (24)$$

The negative sign indicates that work is done by the external force to overcome the electrostatic attraction between the charges.

Exercise 4 - Energy of a Crystal

Consider a "crystal" formed by an infinite chain of alternating charges (ions) $(+q)$ and $(-q)$, separated by a distance a. Determine the energy required to remove an ion from this "crystal".



Let's consider an ion i with charge, for example, $+q$. Its potential energy is just electrostatic, and it's

$$U_i = qV_{es} \ , \tag{25}$$

where V_{es} is the total electrostatic potential generated by the other charges in the position of the ion i. The nearest neighbors are at distance a and have both negative charge $-q$, and their electrostatic potential is

$$2\frac{-q}{4\pi\epsilon_0}\frac{1}{a} \ . \tag{26}$$

The second neighbors are at a distance of $2a$ and positive charge $2q$. Their electrostatic potential in i is

$$2\frac{q}{4\pi\epsilon_0}\frac{1}{2a} \ . \tag{27}$$

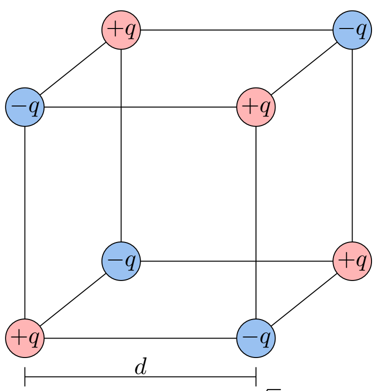
The following neighbors have a charge of opposite sign and distance $3a$ and so on. Summing all the contributions, we find

$$V_{es} = \sum_{n=1}^{\infty} 2\frac{q}{4\pi\epsilon_0}\frac{(-1)^n}{an} = \frac{q}{2\pi\epsilon_0 a}\sum_{n=1}^{\infty}\frac{(-1)^n}{n} = -\frac{q}{2\pi\epsilon_0 a}\log 2 \tag{28}$$

The extraction energy is

$$W = -U_i = \frac{q^2}{2\pi\epsilon_0 a}\log 2 \tag{29}$$

A "unit cell" of an ionic crystal (of the $NaCl$ type) comprises eight point charges at the vertices of a cube with side d. The charges of the ions are $+q$ and $-q$. Calculate the potential energy of this "unit cell."



We start by computing the potential energy of the charge in the lower left corner. The three charges at a distance d have charge $-q$. The electrostatic potential due to them on the position of the first charge is

$$-3\frac{q}{4\pi\epsilon_0}\frac{1}{d} \ . \tag{30}$$

The three charges at a distance $d\sqrt{2}$ have charge q. The electrostatic potential due to them on the position of the first charge is

$$3\frac{q}{4\pi\epsilon_0}\frac{1}{d\sqrt{2}} \ . \tag{31}$$

The last most distant charge is on the opposite corner of the cube at a distance $d\sqrt{3}$ with charge $-q$. The electrostatic potential due to it on the position of the first charge is

$$-\frac{q}{4\pi\epsilon_0}\frac{1}{d\sqrt{3}} \ . \tag{32}$$

The electrostatic potential energy of charge in the bottom left corner is

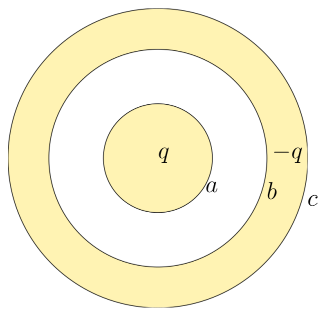
$$U_1 = \frac{q^2}{4\pi\epsilon_0}\frac{1}{d}\left(-3+\frac{3}{\sqrt{2}}-\frac{1}{\sqrt{3}}\right) \ . \tag{33}$$

The total potential energy of the system is four times the potential energy of one charge (to avoid double counting)

$$U = 4\frac{q^2}{4\pi\epsilon_0}\frac{1}{d}\left(-3+\frac{3}{\sqrt{2}}-\frac{1}{\sqrt{3}}\right) \ . \tag{34}$$

Exercise 5 - Energy stored in a condenser

A solid conducting sphere, with radius a and charge $+q$, is placed at the center of a second hollow conducting sphere, with inner radius b and outer radius c. The hollow sphere has a charge of $-q$. Calculate the potential energy stored in this capacitor using the system's capacitance.



We already solved similar exercises in TD 2. We start by computing the electric field everywhere using the Gauss theorem. The result is

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\vec{e}_r \quad \text{for } a < r < b\,, \tag{35}$$

and 0 elsewhere. We set $V(\infty) = 0$ and $V(r) = 0$ for all $r > b$ (because of continuity and $V = constant$ in a conductor). From the definition of the electrostatic field from the potential, we find

$$V(r) = \int_b^r \frac{\mathrm{d}V(r)}{\mathrm{d}r}\mathrm{d}r = -\frac{q}{4\pi\epsilon_0}\int_b^r \frac{1}{r^2}\mathrm{d}r = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{b}\right)\,. \tag{36}$$

The potential inside the internal conductor $r < a$ is again constant and equal to $V(r) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$ for continuity. The capacity of the conductor is

$$C = \frac{Q}{\Delta V} = \frac{q}{\frac{q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)} = \frac{4\pi\epsilon_0}{\frac{1}{a} - \frac{1}{b}}\,. \tag{37}$$

The potential energy of the capacitor can be computed in many equivalent ways

$$\boxed{U = \frac{1}{2}C\Delta V^2 = \frac{Q^2}{2C} = \frac{1}{2}Q\Delta V\,.} \tag{38}$$

In our case

$$U = \frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right) \tag{39}$$

Calculate the potential energy stored in this capacitor by integrating the electric energy density.

The electric energy density of the electrostatic field is given by:

$$\boxed{u = \frac{1}{2}\epsilon_0\vec{E}^2} \tag{40}$$

Substituting the expression for $\vec{E}$ into u, we get:

$$u = \frac{1}{2}\epsilon_0\left(\frac{q}{4\pi\epsilon_0 r^2}\right)^2 \tag{41}$$

The total potential energy U stored in the electric field is obtained by integrating the energy density over the volume between the two spheres:

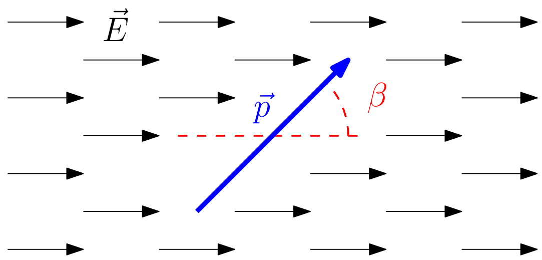
$$U = \int_{Vol} u\,\mathrm{d}V \tag{42}$$

For a spherical shell, the volume element $\mathrm{d}V$ is given by $4\pi r^2\mathrm{d}r$, and the limits of integration are from a to b:

$$U = \int_a^b \frac{1}{2}\epsilon_0\left(\frac{q}{4\pi\epsilon_0 r^2}\right)^2 4\pi r^2\mathrm{d}r = \frac{q^2}{8\pi\epsilon_0}\int_a^b \frac{1}{r^2}\mathrm{d}r = \frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)\,. \tag{43}$$

Exercise 6 - Dipole in a Uniform Field

A dipole is immersed in a uniform electric field $\vec{E}_{ext}$. Its dipole moment $\vec{p}$ makes an angle β with $\vec{E}_{ext}$. Calculate the total force exerted on the dipole.



We will consider the dipole a rigid body. The total force on the dipole is zero because the force on the positive charge will always be equal and in the opposite direction from the force on the negative charge.

Compute the torque $\vec{\tau}$ on the dipole. What is the position of equilibrium?

We compute the torque with respect to the center of the dipole. Using the definition

$$\boxed{\vec{\tau} = \vec{r} \times \vec{F}\ ,} \tag{44}$$

where $\vec{r}$ is the vector from the point we are using as reference to the point where the force $\vec{F}$ is applied. The total torque is the sum of the two contributions to the two forces experienced by the two electric charges of the dipole. We decompose $\vec{p} = q\vec{a}$

$$\vec{\tau} = \frac{\vec{a}}{2} \times (q\vec{E}) + (-\frac{\vec{a}}{2}) \times (-q\vec{E}) = q\vec{a} \times \vec{E} = \vec{p} \times \vec{E}\ . \tag{45}$$

The equilibrium position has $\vec{\tau} = 0$ which means $\vec{p}$ parallel to $\vec{E}$.

Deduce the work provided to the dipole to rotate it by an angle β from its equilibrium position. Then, express the interaction energy U (potential energy) with the surrounding field. For which values of β it is extremal?

The work done by the torque of a force is

$$W = \int \vec{\tau} \cdot \mathrm{d}\vec{\theta} = \int_0^\beta pE \sin\theta \mathrm{d}\theta = pE(1 - \cos\beta) \tag{46}$$

The work is related to the variation of potential energy, therefore

$$U = -pE\cos\beta\ . \tag{47}$$

It is extremal for $\beta = 0, \pi$ when the dipole is aligned with the electrostatic field.