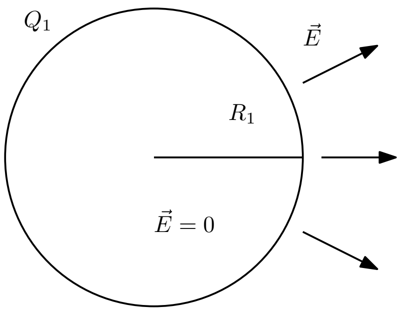
Exercise 1 - Charged conducting spheres and point effect

A sphere S_1, perfectly conductive, with radius R_1, carries a charge Q_1; it is placed in a vacuum. Express the electrostatic field at the surface of the sphere S_1 as a function of Q_1, R_1, and ϵ_0 then with σ_1, the surface charge density, and ϵ_0. What is the distribution of charges?



A conductor is an object or material that allows the flow of charge (electric current) in one or more directions. In electrostatics, statics means that there is no movement of charges, the electric field inside a conductor is zero because the free electrons distribute themselves to cancel out any external electric field. Therefore, we will characterize conductors as objects such that

$$\boxed{\vec{E}_{int} = 0 \text{ inside the conductor .}} \tag{1}$$

We just need to compute the electrostatic field outside the conducting sphere. The problem has a spherical symmetry, and the electrostatic field is radial. Using the Gauss theorem with concentric spheres of radius $r > R_1$, we find the same electrostatic field of a point particle with charge Q_1.

$$\vec{E}_{ext} = \frac{1}{4\pi\epsilon_0}\frac{Q_1}{r^2}\vec{e}_r \text{ outside the conductor .} \tag{2}$$

The electrostatic field right outside the conductor is the limit

$$\lim_{r\to R_1^+} \vec{E}(r) = \frac{1}{4\pi\epsilon_0}\frac{Q_1}{R_1^2}\vec{e}_r \ . \tag{3}$$

The flux of the electrostatic field across all spherical surfaces concentric to the conductor with $r < R_1$ (inside the conductor) vanishes since the field is 0

$$\Phi = \oint \vec{E}\cdot \mathrm{d}\vec{S} = 0 \ . \tag{4}$$

Using the Gauss theorem, we find that

$$\frac{Q_{int}(r)}{\epsilon_0} = \Phi = 0 \ , \quad \to \quad Q_{int}(r) = 0 \text{ for all } r < R_1 \ . \tag{5}$$

There are no charges inside the sphere. Therefore, all the charges are distributed on the sphere's surface (uniformly because of the spherical symmetry). The surface charge distribution is simply

$$Q_1 = 4\pi R_1^2 \sigma_1 \ . \tag{6}$$

The electrostatic field at the surface in terms of σ_1 is

$$\vec{E}_{sup} = \frac{1}{4\pi\epsilon_0}\frac{4\pi R_1^2\sigma_1}{R_1^2}\vec{e}_r = \frac{\sigma_1}{\epsilon_0}\vec{e}_r \ . \tag{7}$$

This is a general result. The electrostatic field outside a conductor is orthogonal to the surface and proportional to the charge density σ_1 divided by ϵ_0.

Provide the field created throughout space by this charge distribution. Deduce the potential throughout space.

The electrostatic field in the whole space has a discontinuity at the surface of the sphere

$$\vec{E} = \begin{cases} \frac{\sigma_1}{\epsilon_0}\vec{e}_r & r > R_1 \\ 0 & r < R_1 \end{cases} \tag{8}$$

We compute the electrostatic potential outside the sphere

$$\vec{E} = -\vec{\nabla} V(r) = -\frac{\mathrm{d}}{\mathrm{d}r} V(r) \ , \tag{9}$$

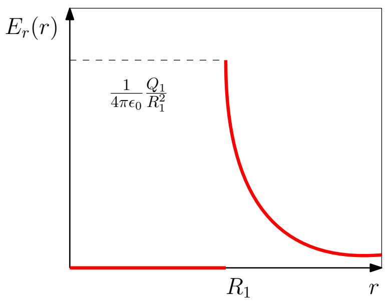
And we can integrate it from ∞ to $r > R_1$ setting $V(\infty) = 0$ conventionally

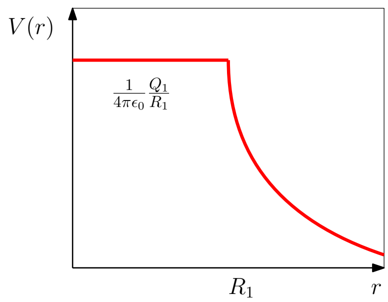
$$-\int_r^{\infty} \frac{\mathrm{d}}{\mathrm{d}r} V(r)\mathrm{d}r = \frac{Q_1}{4\pi\epsilon_0} \int_r^{\infty} \frac{1}{r^2}\mathrm{d}r \quad \rightarrow \quad V(r) = -\frac{Q_1}{4\pi\epsilon_0} \frac{1}{r}\bigg|_r^{\infty} = \frac{Q_1}{4\pi\epsilon_0} \frac{1}{r} . \tag{10}$$

The electrostatic potential at the surface is $V(R_1) = \frac{Q_1}{4\pi\epsilon_0}\frac{1}{R_1}$. The electrostatic potential is constant inside the sphere since the electric field vanishes. We set the constant so that the electrostatic potential is continuous (we have to).

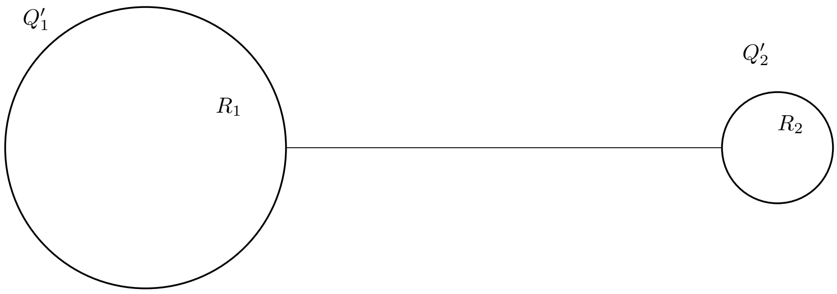
$$V(r) = \begin{cases} \frac{Q_1}{4\pi\epsilon_0}\frac{1}{r} & r > R_1 \\ \frac{Q_1}{4\pi\epsilon_0}\frac{1}{R_1} & r < R_1 \end{cases} \tag{11}$$

We draw the radial component of the electrostatic field and the electrostatic potential as a function of the radial distance





A second conducting sphere, S_2, with a radius $R_2 < R_1$, initially neutral, is now connected by a long and thin conducting wire to the previous sphere S_1. It is assumed that the wire carries no charge and that the influence effects of one sphere on the other are negligible. After connection, the two spheres' charges are denoted Q'_1 and Q'_2. Express Q'_1 and Q'_2 in terms of R_1, R_2, and Q_1.



When we connect the two conductors, the charges from the first conductor are redistributed to the other one. When they reach equilibrium

$$Q_1 = Q'_1 + Q'_2 \qquad \text{the total charge is conserved,} \tag{12}$$

$$V(S_1) = V(S_2) \qquad \text{the two spheres have the same potential,} \tag{13}$$

since they are connected, we can consider them the same conductor. Using the explicit form for the electrostatic potential of a conductive sphere (**??**), we have to solve the system

$$\begin{cases} Q_1 = Q'_1 + Q'_2 \\ \frac{Q'_1}{4\pi\epsilon_0}\frac{1}{R_1} = \frac{Q'_2}{4\pi\epsilon_0}\frac{1}{R_2} \end{cases} \rightarrow \begin{cases} Q_1 = Q'_1 + Q'_2 \\ Q'_1 = \frac{R_1}{R_2}Q'_2 \end{cases} \rightarrow \begin{cases} Q'_1 = \frac{R_1}{R_1+R_2}Q_1 \\ Q'_2 = \frac{R_2}{R_1+R_2}Q_1 \end{cases} \tag{14}$$

The electrostatic potential of the two spheres is

$$V(S_1) = V(S_2) = \frac{Q_1}{4\pi\epsilon_0}\frac{1}{R_1 + R_2} \tag{15}$$

Calculate the fields $\vec{E}_1$ and $\vec{E}_2$ at the surface of the two spheres; deduce a relationship between the ratio $\frac{||\vec{E}_1||}{||\vec{E}_2||}$ and the ratio R_1/R_2. Perform the numerical calculation with $R_1 = 3R_2$. What can be concluded?

The electric field on the surface of the spheres is orthogonal to the surfaces, and using (**??**), we find

$$E(S_1) = \frac{1}{4\pi\epsilon_0}\frac{Q_1'}{R_1^2} = \frac{1}{4\pi\epsilon_0}\frac{Q_1}{R_1}\frac{1}{R_1+R_2} \qquad E(S_2) = \frac{1}{4\pi\epsilon_0}\frac{Q_2'}{R_2^2} = \frac{1}{4\pi\epsilon_0}\frac{Q_1}{R_2}\frac{1}{R_1+R_2}\ . \tag{16}$$

The ratio of the two electrostatic fields is

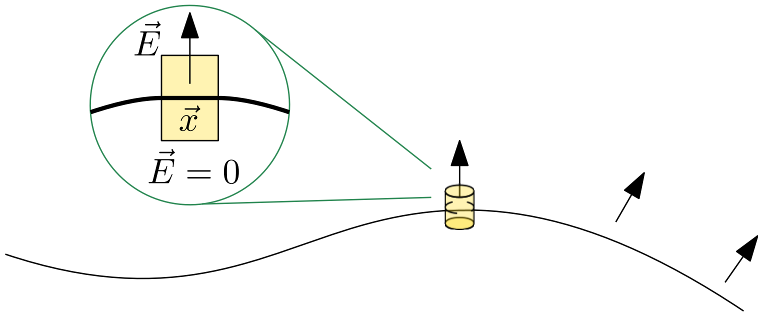
$$\frac{||\vec{E}_1||}{||\vec{E}_2||} = \frac{E(S_1)}{E(S_2)} = \frac{R_2}{R_1} \tag{17}$$

If $R_1 = 3R_2$ the electrostatic field $E(S_1) = \frac{1}{3}E(S_2)$. The electrostatic field of the smaller sphere is three times larger than that of the larger sphere.

Exercise 1.5 - Superficial charge

Consider a conductor with a generic shape. Prove that the electrostatic field at the conductor's surface is orthogonal to the surface and that the surface charge $\sigma(x) = \epsilon_0 E_\perp$.

We use the properties of conductors in electrostatic equilibrium. The electrostatic field is zero inside a conductor because free charges redistribute themselves to cancel any internal electric fields. At the surface, the electrostatic field must be perpendicular to the surface to prevent further movement of charges, ensuring equilibrium. **This means the field is orthogonal to the surface.**

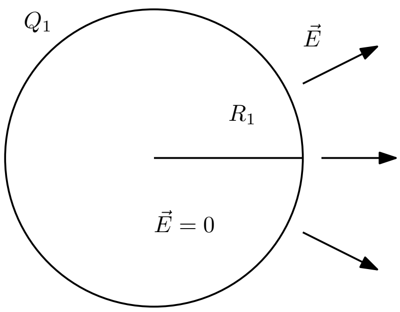


Applying Gauss's law with an infinitesimally small Gaussian surface that straddles the conductor's surface. The electrostatic flux divides into a contribution outside the conductor, inside the conductor, and lateral. The contribution to the flux outside the conductor is $E_\perp \mathrm{d}S$. The flux contribution inside the conductor vanishes since the electric field is zero. The lateral flux contribution is zero because the surface is orthogonal to the field.

$$\Phi = E_\perp(\vec{x})\mathrm{d}S = \frac{Q_{int}}{\epsilon_0} = \frac{\sigma(\vec{x})\mathrm{d}S}{\epsilon_0} \qquad \rightarrow \qquad \sigma(\vec{x}) = \epsilon_0 E_\perp(\vec{x}) \tag{18}$$

Exercise 2 - Charged sphere

A solid conducting sphere, with radius a and charge q, is placed at the center of a second hollow conducting sphere, with inner radius b and outer radius c. The hollow sphere is uncharged. Determine the electric field throughout space and plot it.



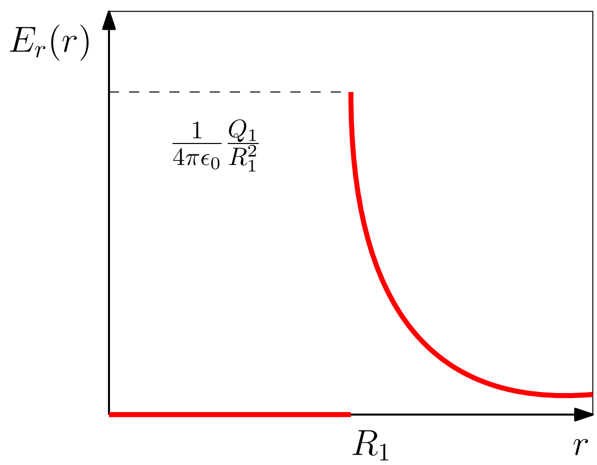
The electrostatic field is radial because of the spherical symmetry and invariances of the problem. By the definition of the conductor, the electrostatic field inside the two conductors is 0. Otherwise, using a spherical surface of radius r, we can prove that the electrostatic field is the same as a point charge q.

$$\vec{E} = \begin{cases} 0 & r < a \\ \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\vec{e}_r & a < r < b \\ 0 & b < r < c \\ \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\vec{e}_r & r > c \end{cases} \tag{19}$$

The electrostatic field has three discontinuities corresponding to the three conductive surfaces.

As we will see in the next point, each discontinuity is associated with a superficial density of charge.

The electrostatic field has the following behavior

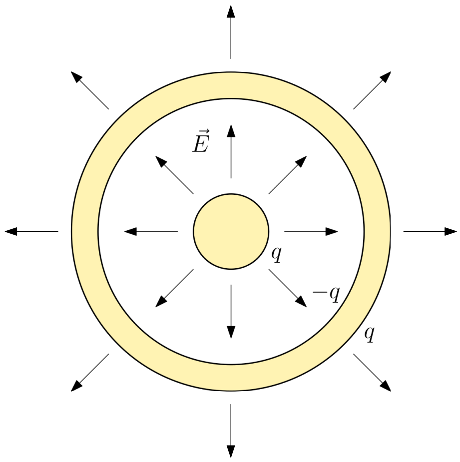


What is the charge on the inner surface of the hollow sphere? What is the charge on the outer surface?

We use the Gauss theorem inside the hollow sphere conductor at a radius $r = b + \delta$ with δ infinitesimally small. The electrostatic field is vanishing. Therefore, the flux of the electrostatic field is also vanishing. Since $Q_{int} = 0$ and we know that the inner sphere has charge q, the inner surface has charge $-q$. By symmetry and by the definition of conductor, the outer surface has a charge q since the hollow sphere has a total charge 0.

Draw the field lines of the system considering the charge q as positive.

The electrostatic field is radial and only outside the conductors.



Calculate the potential at $r = 0$, $r = a$, $r = b$, and $r = c$. The potential reference is considered at infinity.

We compute the potential starting from outside the conductors, integrating from infinity, and using the relation between the radial electrostatic field and the electrostatic potential. For $r > c$ we have

$$-\frac{\mathrm{d}V(r)}{\mathrm{d}r} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} \to V(r) = \frac{1}{4\pi\epsilon_0}\frac{q}{r} \; . \tag{20}$$

Inside the conductor, for $b < r < c$, the electrostatic potential is constant, and by continuity $V(r) = \frac{1}{4\pi\epsilon_0}\frac{q}{c}$. Between the two conductors, for $a < r < b$, we have that

$$V(r) = \frac{1}{4\pi\epsilon_0}\frac{q}{r} + C \; , \tag{21}$$

where we must fix the constant C using the continuity of V in $r = b$

$$C = -\frac{1}{4\pi\epsilon_0}\frac{q}{b} + V(b) = \frac{1}{4\pi\epsilon_0} q \left(-\frac{1}{b} + \frac{1}{c}\right) \; , \tag{22}$$

since $V(b) = V(c)$. Finally, the potential inside the smaller conductor is constant and by continuity $V(r) = \frac{1}{4\pi\epsilon_0}\frac{q}{a} + C$ for $r < a$. To summarize,

$$V(r) = \frac{q}{4\pi\epsilon_0}\begin{cases} \frac{1}{a} - \frac{1}{b} + \frac{1}{c} & r < a \\ \frac{1}{r} - \frac{1}{b} + \frac{1}{c} & a < r < b \\ \frac{1}{c} & b < r < c \\ \frac{1}{r} & r > c \end{cases} \tag{23}$$

Now consider that the charge of the inner sphere is $+q$, and the charge of the second sphere is $-q$. Determine the electric field between the two spheres and outside the second sphere ($r > c$).

The analysis is exactly the same as in the previous case. The main difference is that the electrostatic field for $r > c$ is now vanishing as the two conductors' total charge is 0.

$$\vec{E} = \begin{cases} 0 & r < a \\ \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\vec{e}_r & a < r < b \\ 0 & b < r < c \\ 0 & r > c \end{cases} \tag{24}$$

There is no more superficial density charge at $r = c$.

Calculate the potential $V(r)$.

Also, in this case, the derivation is the same as in the previous case, but the potential outside both conductors is constant (the same as at infinity).

$$V(r) = \frac{q}{4\pi\epsilon_0}\begin{cases} \frac{1}{a} - \frac{1}{b} & r < a \\ \frac{1}{r} - \frac{1}{b} & a < r < b \\ 0 & b < r < c \\ 0 & r > c \end{cases} \tag{25}$$

Draw the electric field lines and equipotential surfaces between the spheres.

The equipotential surfaces are concentric spheres, and the electric field is radial, as in the previous point.

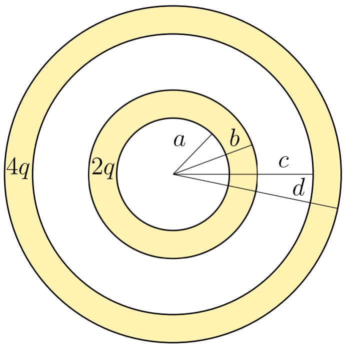
Calculate the capacitance of this system.

The definition of capacitance is the unit charge by unit potential

$$C = \frac{q}{V(a) - V(b)} = 4\pi\epsilon_0 q\frac{ab}{b-a} \tag{26}$$

Exercise 3 - A Matryoshka of Conducting Spheres

A hollow conducting sphere, with an inner radius a and outer radius b, is located inside another hollow conducting sphere with an inner radius c and outer radius d. The two spheres are concentric. The inner sphere has a total charge of $+2q$, and the outer sphere has a total charge of $+4q$. What is the charge on the inner surface of the inner sphere? What is the charge on the outer surface of the inner sphere? The same questions apply to the outer sphere. Determine the electric field in space. Plot the graph E(r).

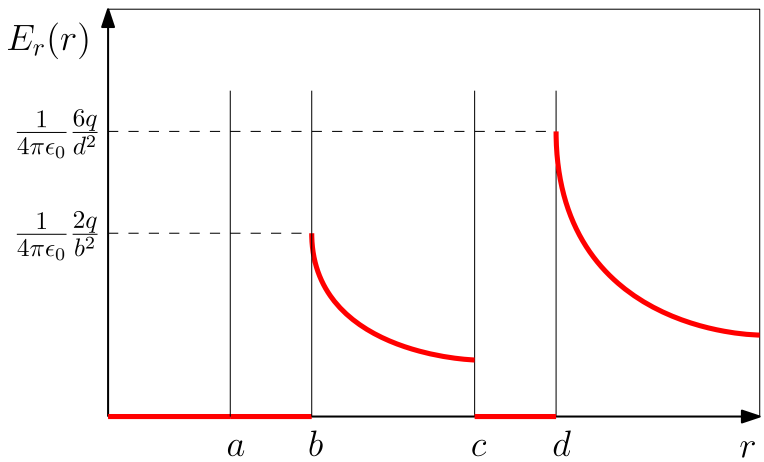


The electrostatic field is radial because of the spherical symmetry and the invariances of the system. We consider a concentric spherical surface inside the smallest hollow sphere of radius $r < a$. The flux of the electrostatic field through this surface is vanishing since there are no charges for $r < a$. The electrostatic field for $r < a$ is also vanishing then. The field for $a < r < b$ and $c < r < d$ vanishes since we are inside a conductor. The field for $b < r < c$ is given by $\frac{1}{4\pi\epsilon_0}\frac{2q}{r^2}\vec{e}_r$. The field outside all the conductors $r > d$ is $\frac{1}{4\pi\epsilon_0}\frac{6q}{r^2}\vec{e}_r$ since the total charge of the two conductors is $6q$. To summarize

$$\vec{E} = \begin{cases} 0 & r < a \\ 0 & a < r < b \\ \frac{1}{4\pi\epsilon_0}\frac{2q}{r^2}\vec{e}_r & b < r < c \\ 0 & c < r < d \\ \frac{1}{4\pi\epsilon_0}\frac{6q}{r^2}\vec{e}_r & r > d \end{cases} \tag{27}$$

The charge on the inner surface of the inner sphere is 0. We can apply the Gauss theorem on a spherical surface with radius $r = a + \delta$ with δ infinitesimal. Since $\vec{E} = 0$, its flux is 0, and the charge is also 0. By charge conservation, the charge on the outer surface of the inner sphere is $2q$.

The charge on the inner surface of the outer sphere is $-2q$. We can apply the Gauss theorem on a spherical surface with radius $r = c + \delta$ with δ infinitesimal. Since $\vec{E} = 0$, its flux is 0, and the internal charge is 0. Since the internal conductor has a charge of $2q$, the charge on the inner surface of the outer conductor has to compensate for it. By charge conservation, the charge on the outer surface of the outer sphere is $6q$. The electrostatic field has the form



Exercise 4 - Capacity of a coaxial cable

A coaxial cable consists of two long concentric conductive cylinders. The outer cylinder, hollow, with an inner radius b, is charged with a linear charge of $-\lambda$. The inner cylinder, solid, with a radius $a < b$, is charged with λ. Calculate the capacitance of this system considering $V = 0$ at $r = b$.

The cylindrical symmetry of the problem implies that the electric field $\vec{E}$ points radially outward from the axis and has a magnitude that depends only on the distance r from the axis. We use Gauss's law to find the electric field at a distance r from the axis, where $a < r < b$. Considering a cylindrical surface concentric to the cable, we find (see the exercise of TD1 for more details)

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0 r}\vec{e}_r \tag{28}$$

The potential difference between two points in an electric field is given by $\Delta V = -\int_a^b \vec{E} \cdot d\vec{r}$.

$$\Delta V = -\int_a^b \frac{\lambda}{2\pi\epsilon_0 r}\mathrm{d}r = -\frac{\lambda}{2\pi\epsilon_0}\log\frac{b}{a} \tag{29}$$

The capacitance of the coaxial cable is given by:

$$C = \frac{Q}{\Delta V} = \frac{\lambda L}{\frac{\lambda}{2\pi\epsilon_0}\log\frac{b}{a}} = \frac{2\pi\epsilon_0 L}{\log\frac{b}{a}} \tag{30}$$

Note that it does not depend on the charge density λ.