Exercise 1 - The Gauss Law

In a region of space, the electric field $\vec{E}$ is uniform. Use Gauss's law to prove that this region must be electrically neutral.

If $\vec{E}$ is uniform we can write it as

$$\vec{E} = E_x \vec{e}_x + E_y \vec{e}_y + E_z \vec{e}_z \ , \tag{1}$$

with E_x, E_y, and E_z constant. We can use the local version of the Gauss theorem. The divergence of the electrostatic field is related to the charge density via

$$\boxed{\vec{\nabla} \cdot \vec{E}(x) = \frac{\rho(x)}{\epsilon_0} \ .} \tag{2}$$

The divergence of the electrostatic field is

$$\vec{\nabla} \cdot \vec{E}(x) = \partial_x E_x + \partial_y E_y + \partial_z E_z = 0 + 0 + 0 = 0 \ , \tag{3}$$

since the components of $\vec{E}$ are all constant. Therefore, $\rho = 0$ in the region where $\vec{E}$ is uniform.

Is the converse true, meaning in a region of space where there is no charge must the electric field be uniform?

This is false. The Gauss theorem states that the **flux** of the electrostatic field depends only on the changes in a region of space. Therefore, the flux of the electrostatic field will vanish in that region. But we cannot say anything about the electrostatic field itself. In general $\vec{E}$ depends on both the charges inside and **outside** the region. Consider the example of the electrostatic Coulomb field generated by a charge q in the origin

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \vec{e}_r \ . \tag{4}$$

Any region that **don't contain the origin**, has 0 charge inside, vanishing flux of the electrostatic field, but non-vanishing $\vec{E}$.

In a region of space, the charge volume density ρ is uniform and positive. Can $\vec{E}$ be uniform in this region?

No. It will contradict the Local version of the Gauss theorem (2).

Exercise 2 - Linear charge

Consider an infinitely long straight wire carrying a uniform charge density λ. What is the symmetry of the problem? What are the invariances?

The problem's symmetry is cylindrical. The charge distribution is invariant under rotations around the wire, translations along the wire, and reflections on any plane containing the wire or orthogonal to the wire. From the symmetry, we can choose a set of cylindrical coordinates (r, ϕ, z) to study the problem, and we align the z axis with the wire. From the invariances, we find that the electrostatic field has only a radial component, and it can only depend on r

$$\vec{E} = E_r(r)\vec{e}_r \tag{5}$$

To be more precise, this is the only time I will write it explicitly. We start with the most general expression of the electrostatic field in cylindrical coordinates

$$\vec{E} = E_r(r,\phi,z)\vec{e}_r + E_\phi(r,\phi,z)\vec{e}_\phi + E_z(r,\phi,z)\vec{e}_z \tag{6}$$

1. The invariance under rotations around the wire and translations along the wire implies that the components of $\vec{E}$ depend only on r and not ϕ and z

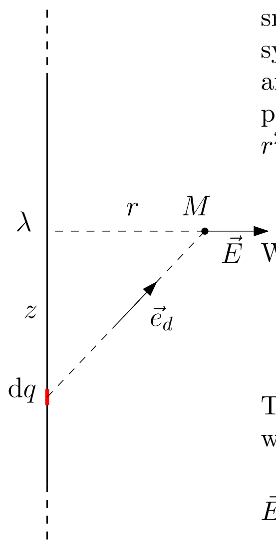
$$E_r(r,\phi,z) = E_r(r) \qquad E_\phi(r,\phi,z) = E_\phi(r) \qquad E_z(r,\phi,z) = E_z(r)\ . \tag{7}$$

2. The invariance under reflections on any plane containing the wire implies $E_\phi(r) = 0$.

3. The invariance under reflections on any plane orthogonal to the wire implies $E_z(r) = 0$.

4. The result is $\vec{E} = E_r(r)\vec{e}_r$

Determine the field $\vec{E}$ created at a point M located at a distance r from the wire starting from the integral expression of the Coulomb electrostatic field.



We start by considering the Coulomb electrostatic field generated from a small piece of wire of length dz of charge d$q = \lambda$dz. Using the problem's symmetries, we can assume the coordinate of the point $M = (r, \phi = 0, z = 0)$ and the small piece of wire at the coordinate $(r = 0, \phi = 0, z)$. The Coulomb potential generated by dq in the point M is in the direction $\vec{e}_d$ with $d^2 = r^2 + z^2$

$$\vec{E}_{\mathrm{d}q} = \frac{1}{4\pi\epsilon_0}\frac{\mathrm{d}q}{d^2}\vec{e}_d = \frac{\lambda}{4\pi\epsilon_0}\frac{\mathrm{d}z}{r^2+z^2}\vec{e}_d\ . \tag{8}$$

We can project the $\vec{E}_{\mathrm{d}q}$ on cylindrical coordinates (we use trigonometry)

$$\vec{E}_{\mathrm{d}q} = \frac{\lambda}{4\pi\epsilon_0}\frac{\mathrm{d}z}{r^2+z^2}\frac{r}{\sqrt{r^2+z^2}}\vec{e}_r + \frac{\lambda}{4\pi\epsilon_0}\frac{\mathrm{d}z}{r^2+z^2}\frac{z}{\sqrt{r^2+z^2}}\vec{e}_z\ . \tag{9}$$

The total electrostatic field is the result of summing $\vec{E}_{\mathrm{d}q}$ of all the pieces of wire (integrating)

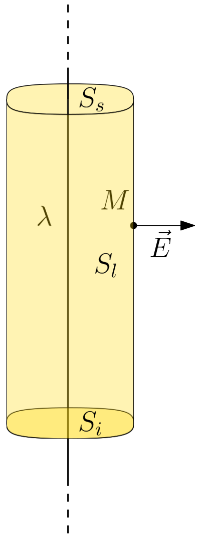
$$\vec{E} = \int_{-\infty}^{\infty}\vec{E}_{\mathrm{d}q} = \frac{\lambda}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{\mathrm{d}z}{r^2+z^2}\frac{r}{\sqrt{r^2+z^2}}\vec{e}_r + \frac{\lambda}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{\mathrm{d}z}{r^2+z^2}\frac{z}{\sqrt{r^2+z^2}}\vec{e}_z\ . \tag{10}$$

We know that because of the invariances of the problem, the $\vec{e}_z$ component has to vanish (you can also prove it analytically since it is the integral of an odd function of z on a symmetric interval). To compute the integral on the radial part, we make a change of variable $z = r\zeta$ and find

$$\vec{E} = \frac{\lambda}{4\pi\epsilon_0}\frac{1}{r}\int_{-\infty}^{\infty}\frac{\mathrm{d}\zeta}{(1+\zeta^2)^{3/2}}\vec{e}_r = \frac{\lambda}{2\pi\epsilon_0}\frac{1}{r}\vec{e}_r\ . \tag{11}$$

Note that the integral $\int_{-\infty}^{\infty} \frac{\mathrm{d}\zeta}{(1+\zeta^2)^{3/2}} = \left.\frac{\zeta}{\sqrt{1+\zeta^2}}\right|_{-\infty}^{\infty} = 2$.

Determine the field $\vec{E}$ created at a point M located at a distance r from the wire using the Gauss theorem.



We use the integral version of the Gauss theorem. The flux of the electrostatic field through a closed surface is proportional to the enclosed charges Q_{int}

$$\boxed{\Phi = \oint \vec{E}\cdot \mathrm{d}\vec{S} = \frac{Q_{int}}{\epsilon_0}\ .} \tag{12}$$

We choose as a closed surface the cylinder of height h with the wire at the center with M at the border. The flux of the electrostatic divides into 3 contributions (the lateral l surface, the superior surface s, and the inferior surface i)

$$\Phi = \Phi_l + \Phi_s + \Phi_i\ . \tag{13}$$

The electrostatic field is radial (from equation (5)). The normal of the superior and inferior flux are $\pm\vec{e}_z$. The scalar product $\vec{E}\cdot \mathrm{d}\vec{S} = 0$ therefore

$$\Phi_s = \Phi_i = 0\ . \tag{14}$$

The lateral flux is

$$\Phi_l = \oint \vec{E}\cdot \mathrm{d}\vec{S} = \oint E_r(r)\vec{e}_r \cdot \mathrm{d}\vec{S} = E_r(r) 2\pi r h\ . \tag{15}$$

The charges inside the surface are $Q_{int} = \lambda h$.

$$E_r(r) 2\pi r h = \frac{\lambda h}{\epsilon_0} \quad\longrightarrow\quad E_r(r) = \frac{\lambda}{2\pi\epsilon_0 r} \tag{16}$$

Deduce the potential $V(r)$ at M.

From the invariances of the problem, we can deduce that V is just a function of the radial coordinate $V(r)$. The electrostatic field is minus the gradient of the electrostatic potential

$$\boxed{\vec{E} = -\vec{\nabla} V\ .} \tag{17}$$

The gradient in cylindrical coordinates is

$$\vec{\nabla} f(r,\phi,z) = \frac{\partial f}{\partial r}\vec{e}_r + \frac{1}{\rho}\frac{\partial f}{\partial \varphi}\vec{e}_\phi + \frac{\partial f}{\partial z}\vec{e}_z\ . \tag{18}$$

Since $V(r)$ depends only on r the derivatives $\frac{\partial V(r)}{\partial \varphi} = \frac{\partial V(r)}{\partial z} = 0$ and the equation (17) is

$$\frac{\mathrm{d}V(r)}{\mathrm{d}r} = -E_r = -\frac{\lambda}{2\pi\epsilon_0 r}\ . \tag{19}$$

We can integrate from a reference r_0 to a general r on both sides.

$$\int_{r_0}^{r} \frac{\mathrm{d}V(r)}{\mathrm{d}r}\mathrm{d}r = -\frac{\lambda}{2\pi\epsilon_0}\int_{r_0}^{r}\frac{1}{r}\mathrm{d}r \tag{20}$$

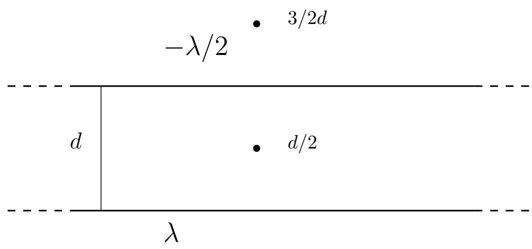
$$V(r) - V(r_0) = -\frac{\lambda}{2\pi\epsilon_0}\log\frac{r}{r_0}\ . \tag{21}$$

We choose conventionally $V(r_0) = 0$ and find

$$V(r) = -\frac{\lambda}{2\pi\epsilon_0}\log\frac{r}{r_0}\ . \tag{22}$$

Remember that the electrostatic potential is defined up to a constant, allowing us to fix $V(r_0) = 0$.

An infinite straight wire aligned along the x axis has a charge per unit length of λ. A second straight wire, parallel to the x axis and located at $y = d$, has a linear charge density of $\lambda_2 = -\lambda/2$. Calculate the electric field at $y = \frac{d}{2}$ and at $y = \frac{3}{2}d$.



The electrostatic fields are additive. Therefore, the electrostatic field for two wires is the sum of the electrostatic fields of the single wires. We set $z = 0$ for simplicity. If $y = d/2$, since the charges of the two wires are opposite, the electrostatic fields point in the same direction, and they sum

$$\vec{E}(y) = \frac{\lambda}{2\pi\epsilon_0 y}\vec{e}_y + \frac{\lambda}{4\pi\epsilon_0(d-y)}\vec{e}_y \qquad \rightarrow \qquad \vec{E}(d/2) = \frac{3\lambda}{2\pi\epsilon_0 d}\vec{e}_y \ . \tag{23}$$

Conversely, if $y > d$, the two electrostatic fields point in opposite directions, and they subtract

$$\vec{E}(y) = \frac{\lambda}{2\pi\epsilon_0 y}\vec{e}_y - \frac{\lambda}{4\pi\epsilon_0(y-d)}\vec{e}_y \qquad \rightarrow \qquad \vec{E}(3d/2) = -\frac{\lambda}{6\pi\epsilon_0 d}\vec{e}_y \ . \tag{24}$$

A long cylinder with a radius R is charged with a uniform charge density λ. What are the equipotential surfaces for this cylinder? Considering that the potential is zero (reference) at the surface of the cylinder, what are the radii of the equipotential surfaces corresponding to V_0, $2V_0$, and $3V_0$, respectively? Are they equally spaced?

The electrostatic potential of a linear charged cylinder is the same as that of a wire (22). To set the potential 0 at $r = R$ we take

$$V(r) = -\frac{\lambda}{2\pi\epsilon_0}\log\frac{r}{R} \ . \tag{25}$$

To find the radius of the surface at potential V, we can invert the equation

$$r = Re^{-\frac{2\pi\epsilon_0 V}{\lambda}} \ . \tag{26}$$

Notice that the dependence is exponential in V. The equipotential surfaces corresponding to V_0, $2V_0$, and $3V_0$ are not equally spaced.

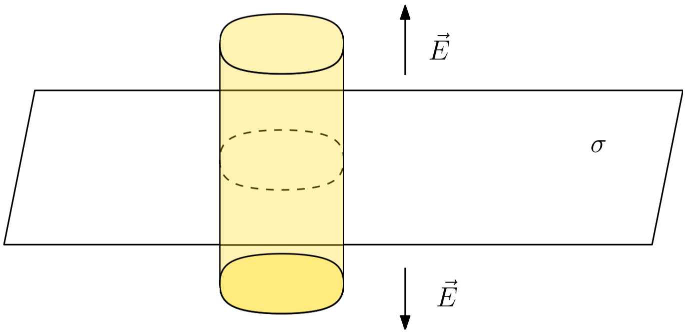
Exercise 4 - Surface charge on a plane

Consider an infinite plane carrying a uniform surface charge density σ. What is the symmetry of the problem? What are the invariances?

The system has a planar symmetry. We have invariance under translation along the plane and reflection with respect to all planes orthogonal to the charged plane and with respect to the charged plane itself. This reduces the electrostatic field in Cartesian coordinates (assuming the charged plane lies on xy) to the form

$$\vec{E} = \begin{cases} E_z(z)\vec{e}_z & z > 0 \\ -E_z(-z)\vec{e}_z & z < 0 \end{cases} \tag{27}$$

Using Gauss's theorem, express the electrostatic field $\vec{E}$ at a distance z on either side of the plane. Express the discontinuity of the field when crossing the plane as a function of σ and ϵ_0.



Consider as a closed surface a cylinder of height h and radius r orthogonal to the plane. The flux of the electrostatic field gets a contribution only from the top and bottom surfaces with equal amounts.

$$\Phi = \pi r^2 E_z(h) - \pi r^2(-E_z(-(-h))) = 2\pi r^2 E_z(h) \tag{28}$$

The charge inside the surface is $Q_{int} = \pi r^2 \sigma$. Therefore, from the Gauss theorem

$$2\pi r^2 E_z(h) = \pi r^2 \frac{\sigma}{\epsilon_0} \,, \qquad \rightarrow \qquad E_z(h) = \frac{\sigma}{2\epsilon_0} \,. \tag{29}$$

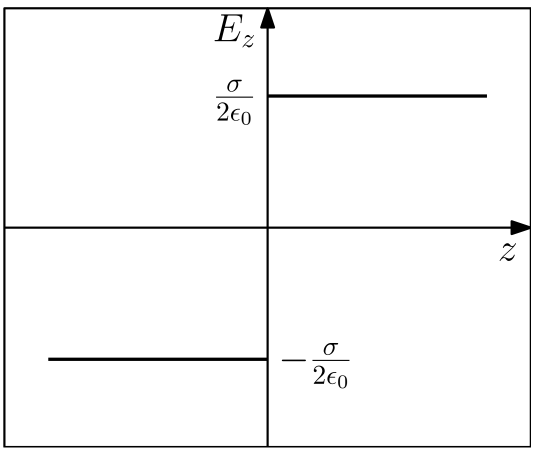
The electrostatic field is

$$\vec{E} = \begin{cases} \frac{\sigma}{2\epsilon_0}\vec{e}_z & z > 0 \\ -\frac{\sigma}{2\epsilon_0}\vec{e}_z & z < 0 \end{cases} \tag{30}$$

The discontinuity is

$$\Delta E_z = E_z(0^+) - E_z(0^-) = \frac{\sigma}{\epsilon_0} \,. \tag{31}$$

The graph of $E_z(z)$ is



Exercise 5 - Two charged planes

Consider two parallel planes at a distance h charged with surface charge σ and $-\sigma$. By exploiting the symmetry of the problem and using Laplace's equation, calculate the potential V at any point between the two planes as a function of h and their potentials V_1 and V_2.

The problem has the same planar symmetry as before. The Laplace equation for the electrostatic potential is

$$\boxed{\vec{\nabla}^2 V = -\frac{\rho}{\epsilon_0}\ .} \tag{32}$$

In this case, the potential depends only on z, and the Laplacian is an ordinary second derivative. The charge density is 0 anywhere within the plates $0 < z < h$ and the Laplace equation reduces to

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2} V(z) = 0\ . \tag{33}$$

The most general function with a vanishing second derivative is $V(z) = c_1 z + c_0$. Imposing the boundary conditions $V(0) = V_1$ and $V(h) = V_2$ we find

$$V(z) = V_0 + \frac{V_1 - V_0}{h} z\ . \tag{34}$$

Provide the vectorial expression of the electrostatic field $\vec{E}$ using $\vec{E} = -\vec{\nabla} V$. Find another expression for the electrostatic field $\vec{E}$ between the two planes using the superposition theorem and the results from exercise 4.

The electrostatic field between the two planes can be immediately obtained by deriving the potential

$$\vec{E} = -\frac{\mathrm{d}}{\mathrm{d}z} V(z)\vec{e}_z = -\frac{V_1 - V_0}{h}\vec{e}_z\ . \tag{35}$$

Superimposing the electrostatic fields of two charged planes with opposite charges, for $0 < z < h$, we find

$$\vec{E} = \frac{\sigma}{2\epsilon_0}\vec{e}_z - \frac{-\sigma}{2\epsilon_0}\vec{e}_z = \frac{\sigma}{\epsilon_0}\vec{e}_z\ . \tag{36}$$

By direct comparison, for this system, we find the relation

$$V_1 - V_0 = -\frac{\sigma h}{\epsilon_0} \tag{37}$$

Derive the expression for the capacity of the capacitor formed by the two charged planes.

By definition, the capacitance is the total charge per unit of electrostatic potential energy

$$C = \frac{Q}{\Delta V} = \frac{S\sigma}{V_0 - V_1} = \frac{S\epsilon_0}{h}\ . \tag{38}$$