Appendix 1 - Explicit solution of the heat equation

The Laplacian in spherical coordinates (r, θ, ϕ) takes the form

$$\nabla^2 T = \frac{1}{r^2}\partial_r\left(r^2\partial_r T\right) + \frac{1}{r^2\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta T\right) + \frac{1}{r^2\sin^2\theta}\partial^2_\phi T \ . \tag{1}$$

Suppose the system has **spherical symmetry**. The temperature field is independent of the angular variables θ and ϕ and depends only on the radial variable r and time t. The temperature diffusion equation becomes

$$\partial_t T(r,t) = \kappa\frac{1}{r^2}\partial_r\left(r^2\partial_r T(r,t)\right) \ . \tag{2}$$

With the following boundary conditions, we want to solve equation (**??**) in a sphere of radius R_0 (representing the potato). We place the sphere in a fixed temperature T_w heat bath, therefore $T(R_0,t) = T_w$. We assume that initially $(t = 0)$, all points of the sphere are at the same temperature T_a (the potato is at room temperature) $T(r,0) = T_a$ for $r < R_0$. We solve equation (**??**) introducing the auxiliary field $B(r,t) = r\left(T(r,t) - T_w\right)$ that satisfy the differential equation

$$\partial_t B(r,t) = \kappa\partial^2_r B(r,t) \ . \tag{3}$$

The field $B(r,t)$ satisfy three boundary conditions

1. The thermal bath condition $B(R_0,t) = R_0\left(T(R_0,t) - T_w\right) = 0$ because $T(R_0,t) = T_w$.
2. The initial temperature condition $B(r,0) = r\left(T(r,0) - T_w\right) = r\left(T_a - T_w\right)$.
3. The regularity condition in the origin $B(0,t) = 0\cdot\left(T(0,t) - T_w\right) = 0$ (the temperature is finite at the center of the sphere).

We solve this equation using the separation of variables $B(r,t) = \rho(r)\tau(t)$ and reorganize (**??**) as

$$\frac{\partial_t\tau(t)}{\tau(t)} = \kappa\frac{\partial^2_\rho\rho(r)}{\rho(r)} = -\kappa\omega^2 \ , \tag{4}$$

where ω is an arbitrary (a priori complex) constant since the first term of the equality is independent of r and the second term is independent of t. The time component is solved immediately as $\tau(t) \propto e^{-\kappa\omega^2 t}$ which imposes $\omega \in \mathbb{R}$ since we want a regular solution for all times. The radial part of the equation $\partial^2_\rho\rho(r) = -\omega^2\rho(r)$ is the standard oscillatory equation that is solved by trigonometric functions sin and cos. The generic solution is of the form

$$B(r,t) = \left(\alpha_\omega\cos\omega r + \beta_\omega\sin\omega r\right)e^{-\kappa\omega^2 t} \tag{5}$$

Imposing the regularity condition, we find that

$$B(0,t) = \alpha_\omega e^{-\kappa\omega^2 t} = 0 \rightarrow \alpha_\omega = 0 \ , \tag{6}$$

while the thermal bath boundary conditions require

$$B(R_0,t) = \beta_\omega\sin\omega R_0 e^{-\kappa\omega^2 t} = 0 \rightarrow \omega R_0 = n\pi \ n \in \mathbb{N} \ , \tag{7}$$

The most general solution is a linear combination of all the n

$$B(r,t) = \sum_{n=1}^{\infty}\beta_n\sin\left(\frac{n\pi}{R_0}r\right)e^{-\kappa\left(\frac{n\pi}{R_0}\right)^2 t} \ . \tag{8}$$

We determine the coefficients β_n imposing the initial temperature boundary condition

$$B(r,0) = \sum_{n=1}^{\infty}\beta_n\sin\left(\frac{n\pi}{R_0}r\right) = r\left(T_a - T_w\right) \ . \tag{9}$$

We multiply by $\sin\left(\frac{m\pi}{R_0}r\right)$ on both sides and integrate in r between 0 and R_0 to find

$$\frac{R_0}{2}\beta_n = (T_a - T_w)\int_0^{R_0} r\sin\left(\frac{n\pi}{R_0}r\right)\mathrm{d}r = (T_w - T_a)\,(-1)^n\frac{R_0^2}{n\pi}\ . \tag{10}$$

Finally, the general solution of the temperature diffusion equation in the spherical potato is

$$\frac{T(r,t) - T_w}{T_w - T_a} = 2\sum_{n=1}^{\infty}(-1)^n\frac{\sin\left(\frac{n\pi}{R_0}r\right)}{\frac{n\pi}{R_0}r}e^{-\kappa\left(\frac{n\pi}{R_0}\right)^2 t}\ . \tag{11}$$