# Quantum info of LQG states

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Collaboration with Eugenio Bianchi
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and w.i.p.

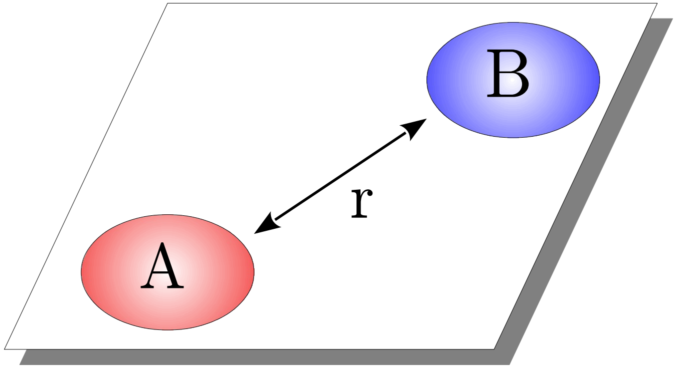
Plan of the talk:

Motivations and background

Typical entropy: Page curve and its variance

Entropy and correlations in Bell-network states

# Correlations at space-like separation



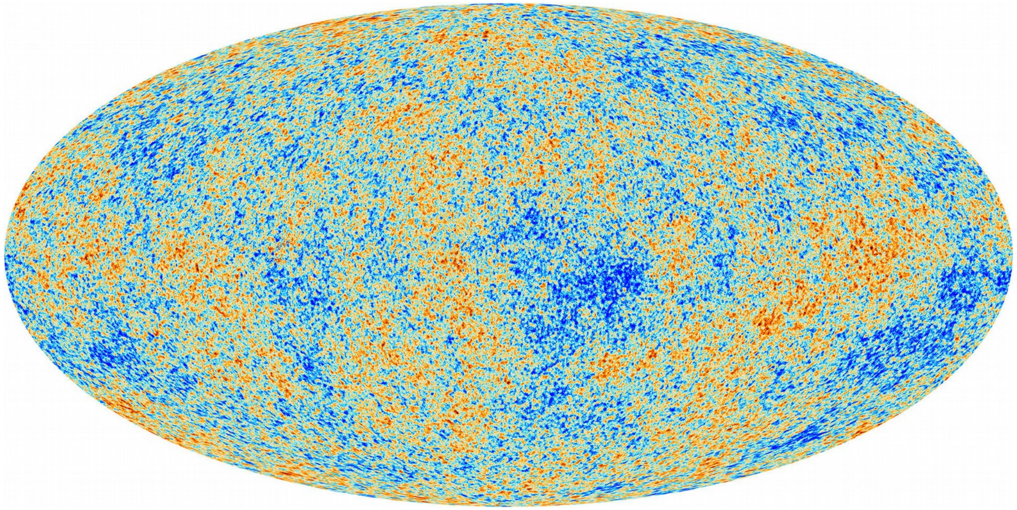

Quantum field theory $\longrightarrow$ Fock space contains:

~~(i) states with no space-like correlations~~

(ii) states *with* specific short-ranged correlations

(e.g. Minkowski vacuum)

$$\langle \phi(0)\phi(r)\rangle \approx r^{-2}$$



Loop quantum gravity

$$\mathcal{H} = \bigoplus_{\Gamma} \bigoplus_{j_\ell} \bigotimes_{n} \mathcal{H}_n$$

(i) states with no space-like correlations (spin-networks)

(ii) states *with* specific short-ranged correlations (many proposals)

# LQG Hilbert space

$$\mathcal{H}_\Gamma = \bigoplus_{j_\ell} \bigotimes_n \mathcal{H}_n$$

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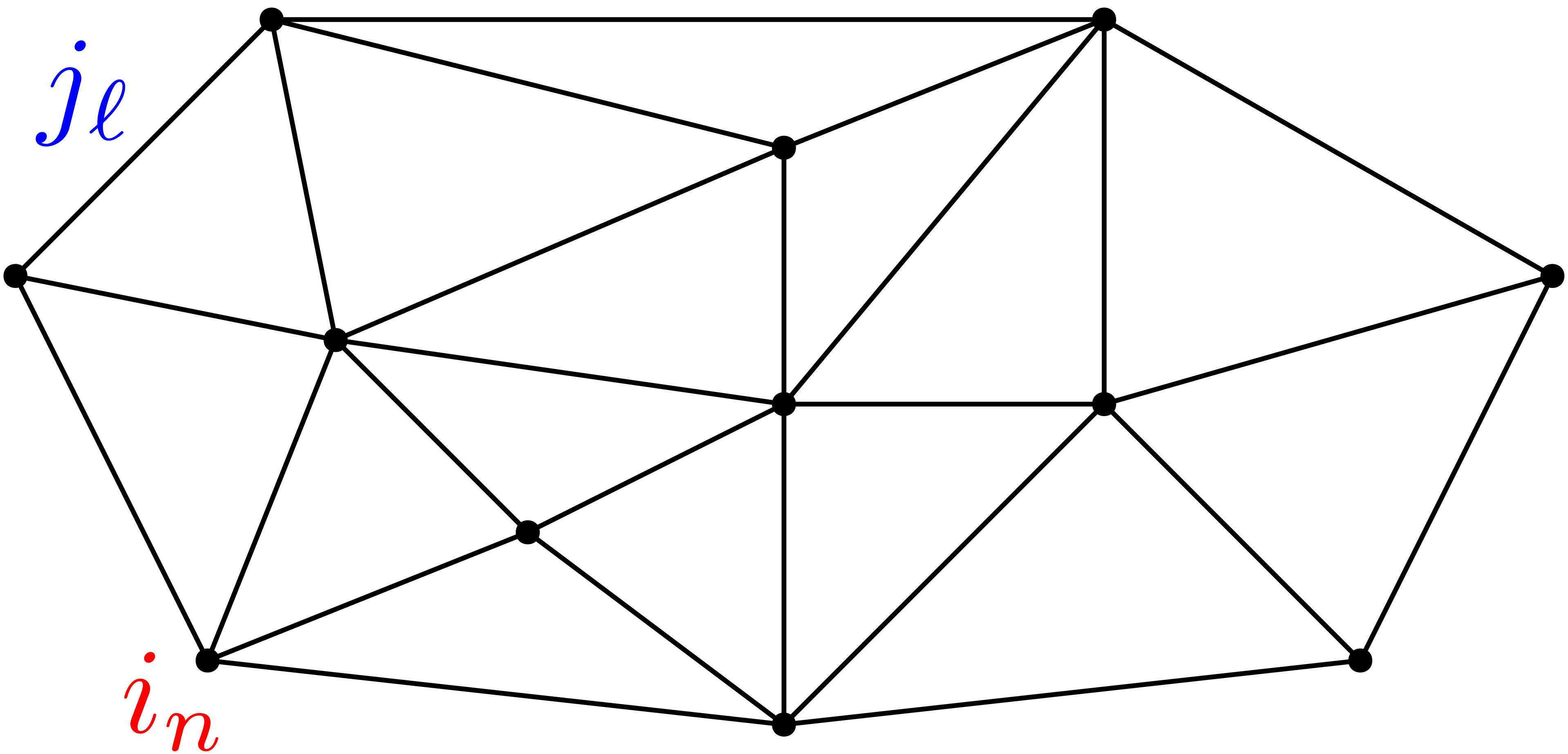
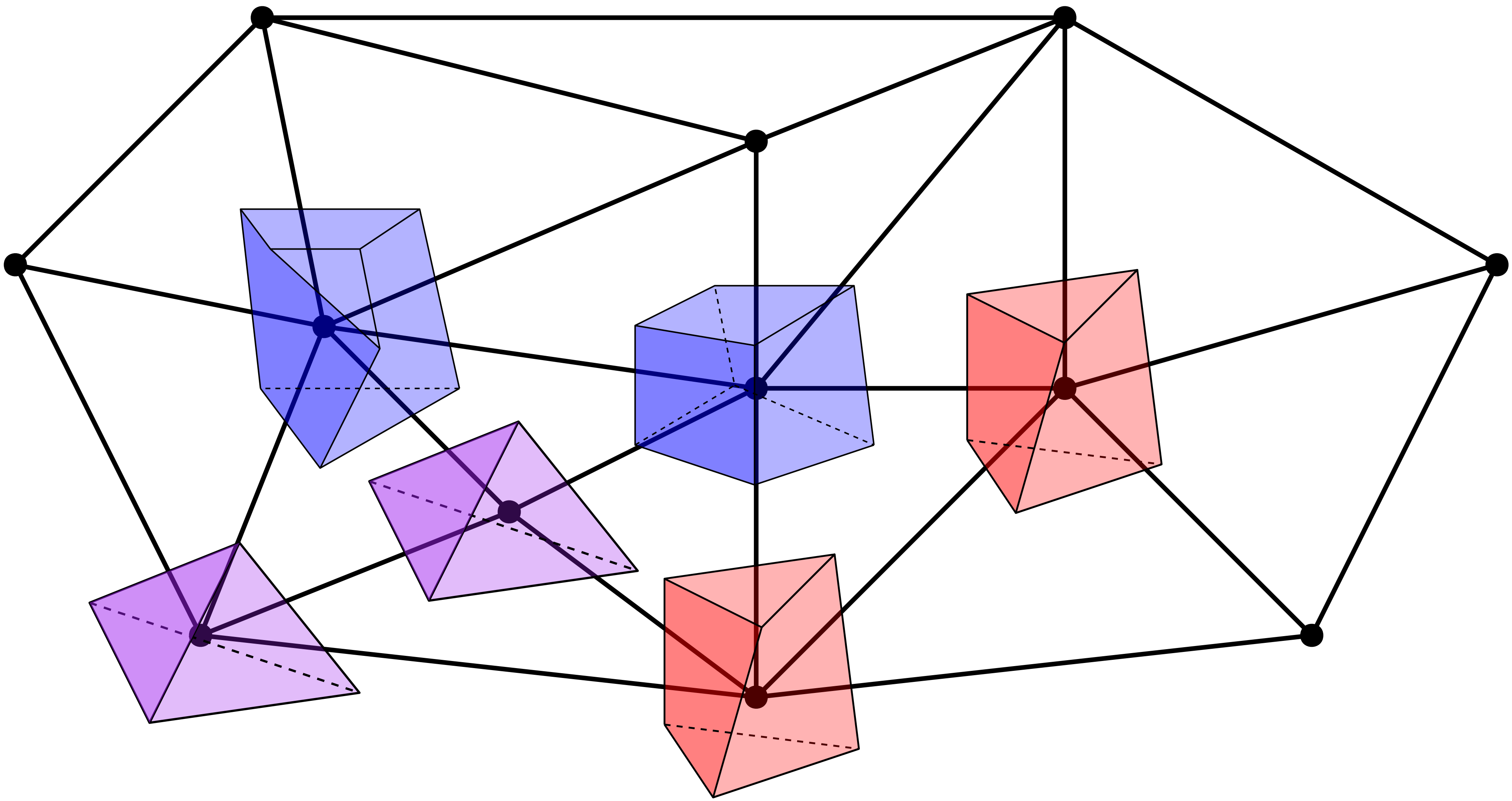



$$\mathcal{H}_n = \mathrm{Inv}\left(j_1 \otimes \cdots \otimes j_F\right)$$

# Quantum Polyhedra

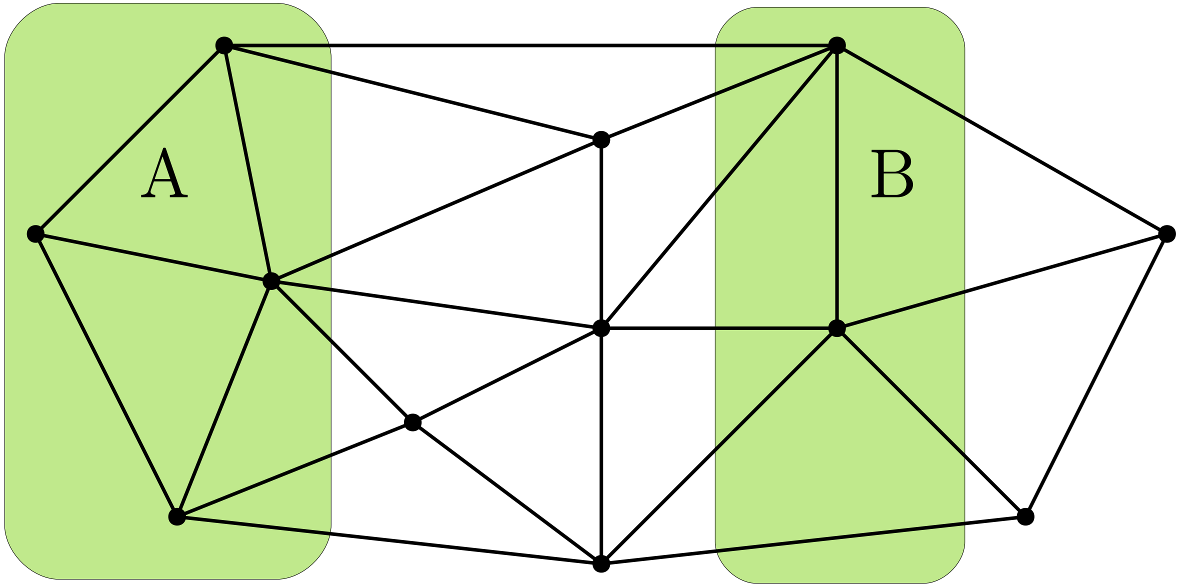
[Bianchi, P.D., Speziale PRD 2010]

$$\mathcal{H}_\Gamma = \bigoplus_{j_\ell} \bigotimes_n \mathcal{H}_n$$



Geometric picture from LQG states

# Information-theoretic bounds on correlations




State

$$|\psi\rangle \in \mathcal{H}_\Gamma$$

Subsystem A

$$O_A \in \mathcal{A}_A$$

Correlations

$$\mathcal{C}\left(O_A, O_B\right) = \langle\psi| O_A O_B |\psi\rangle - \langle\psi| O_A |\psi\rangle \langle\psi| O_B |\psi\rangle$$

Bounded by mutual information

$$\frac{1}{2}\left(\frac{\mathcal{C}\left(O_A . O_B\right)}{\|O_A\|\|O_B\|}\right)^2 \leq S_A(\psi) + S_B(\psi) - S_{AB}(\psi)$$

[Wolf, Verstraete, Hastings, Cirac PRL 2008]

Entropy zero law or volume law = no correlations

What about a random state?

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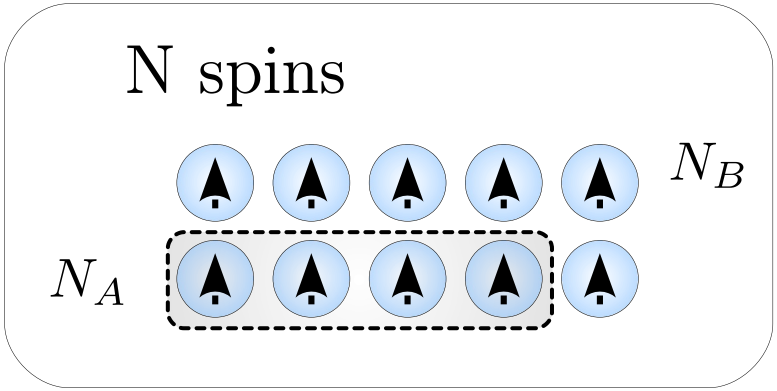
# Typical entropy of a subsystem (with an example)

Hilbert space $\mathcal{H} = \bigotimes_{i=1}^{N} \mathbb{C}^2$ $\qquad d = 2^N$

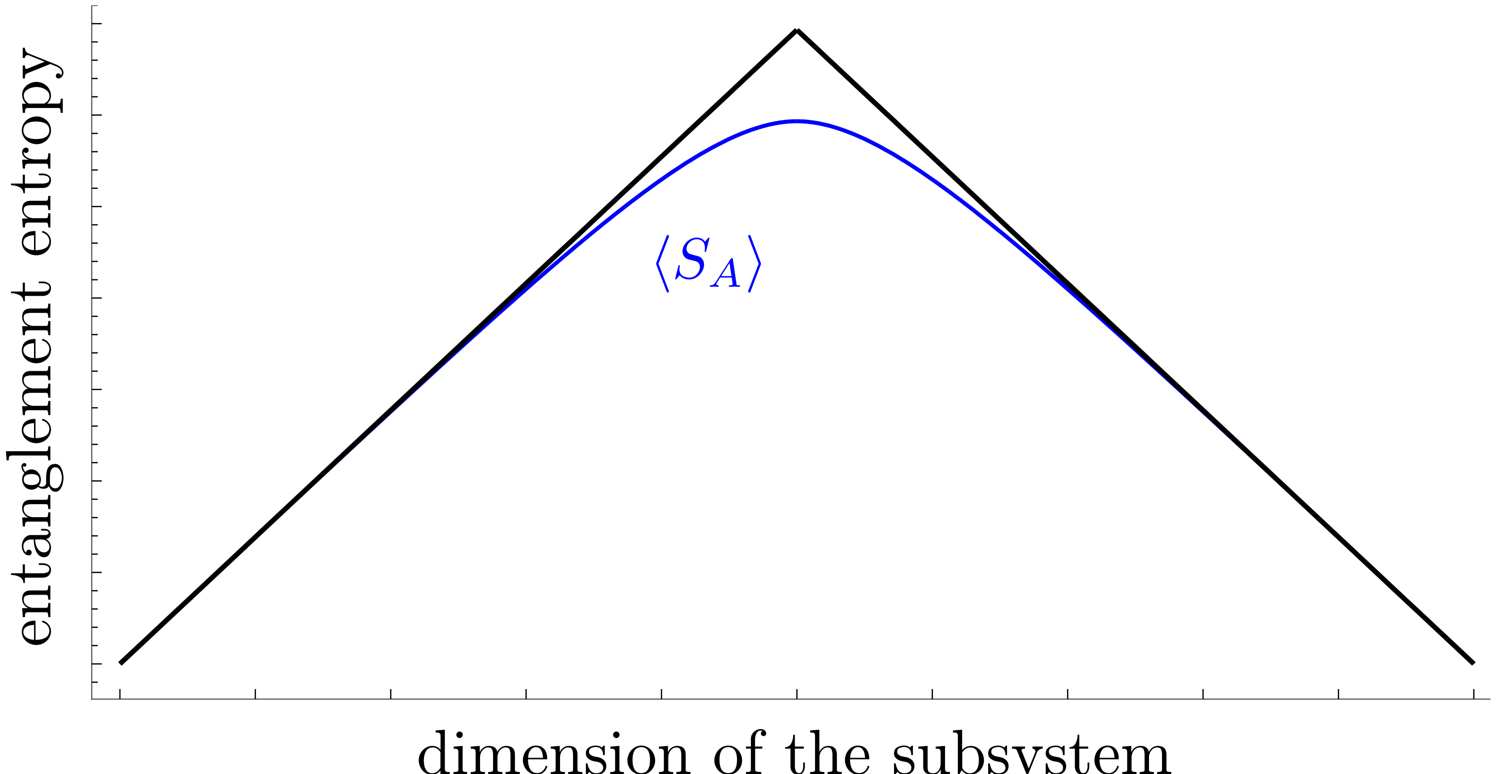



Subsystem $N_A$ spins $\qquad d_A = 2^{N_A}$

$\rho_A = \mathrm{Tr}_B \left( |\psi\rangle \langle\psi| \right) \qquad S_A(\psi) = -\mathrm{Tr}_A \rho_A \log \rho_A$

The average entropy (random uniform) is close to maximal

[Page PRL 1995]



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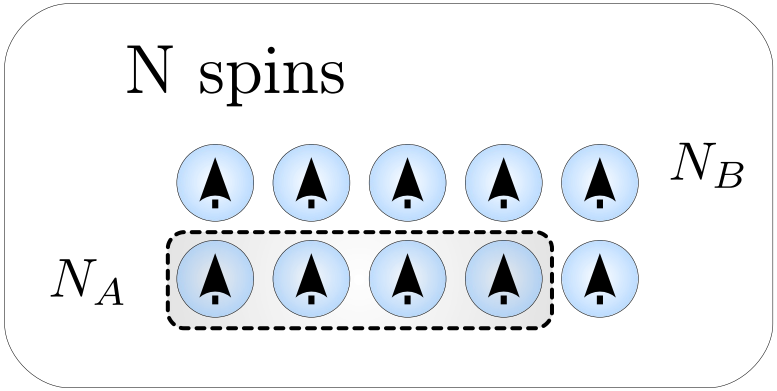


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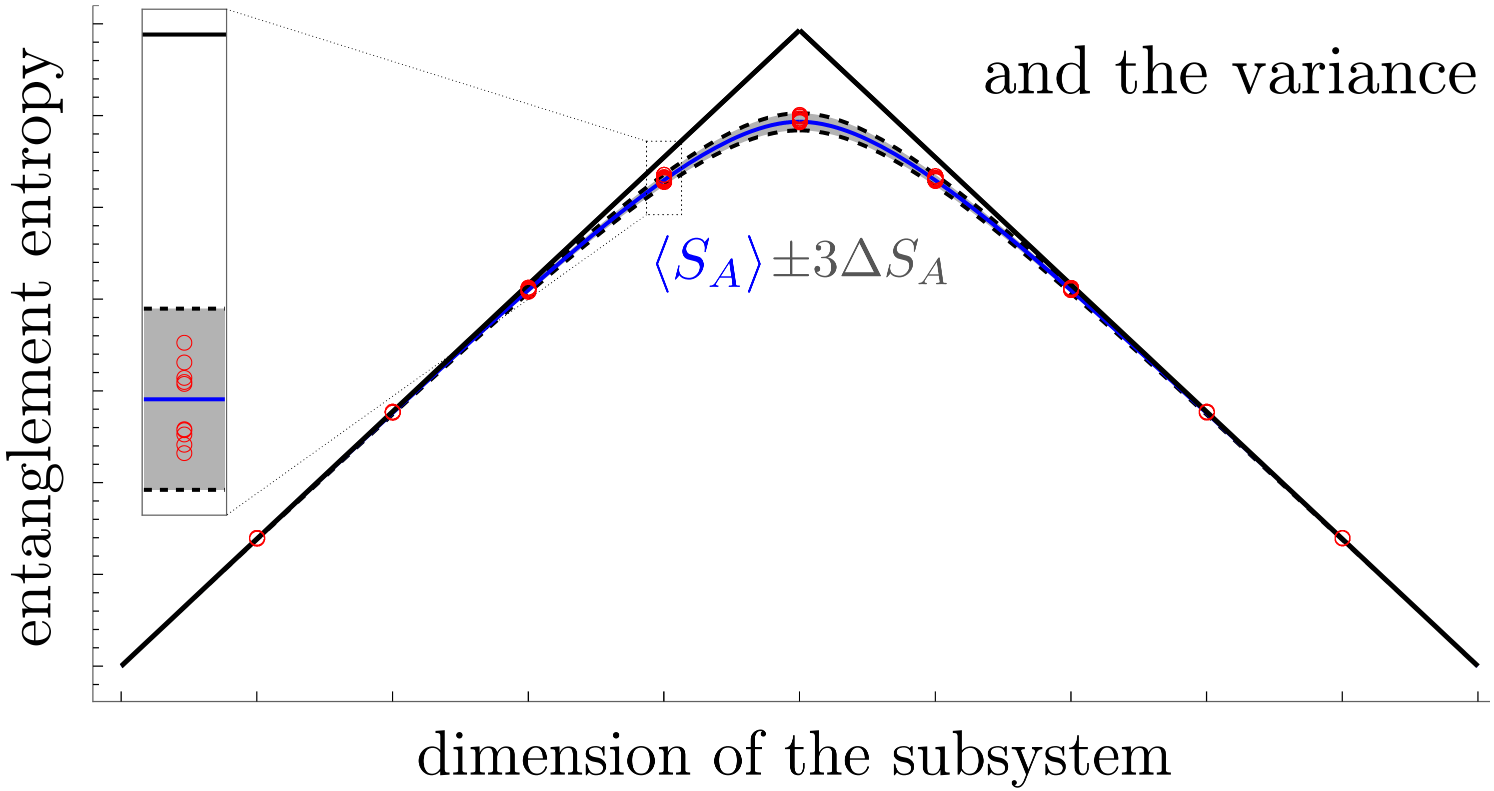
The average entropy (random uniform) is close to maximal

[Page PRL 1995]

and the variance is small

[Bianchi, P.D. PRD 2019]



# Sketch of the proof

[Page PRL 1995, Bianchi, P.D. PRD 2019]

step 1:

$$\langle \mathrm{Tr}\rho_A^r \rangle = \int \mathrm{d}\mu(\psi)\, \mathrm{Tr}\rho_A^r = \int \left( \sum_{a=1}^{d_A} \lambda_a^r \right) \mu(\lambda_1, \ldots, \lambda_{d_A}) \prod_{b=1}^{d_A} \mathrm{d}\lambda_b$$

Eigenvalues of the density matrix
and induced integration measure
[Lloyd, Pagels Ann. Phys 1988]

step 2:

$$\langle S_A \rangle = - \langle \mathrm{Tr}\rho_A \log \rho_A \rangle = - \lim_{r \to 1} \partial_r \, \langle \mathrm{Tr}\rho_A^r \rangle$$

digamma function $\Psi = \Gamma'/\Gamma$

exact result:

$$\langle S_A \rangle = \Psi(d_A d_B + 1) - \Psi(d_B + 1) - \frac{d_A - 1}{2 d_B}$$

asymptotic:

$$\langle S_A \rangle \approx \log d_A - \frac{d_A^2 - 1}{2 d_A d_B} \qquad d_B \gg 1$$

$$d_A \leq d_B$$

# Sketch of the proof

[Bianchi, P.D. PRD 2019]

variance:

$$\Delta S_A^2 = \langle S_A^2 \rangle - \langle S_A \rangle^2$$

step 1:

$$\langle \mathrm{Tr}\rho_A^{r_1} \mathrm{Tr}\rho_A^{r_2} \rangle = \int \left( \sum_{a=1}^{d_A} \lambda_a^{r_1} \right) \left( \sum_{a=1}^{d_A} \lambda_a^{r_2} \right) \mu(\lambda_1, \ldots, \lambda_{d_A}) \prod_{b=1}^{d_A} \mathrm{d}\lambda_b$$

step 2:

$$\langle S_A^2 \rangle = \lim_{\substack{r_1 \to 1 \\ r_2 \to 1}} \partial_{r_1} \partial_{r_2} \langle \mathrm{Tr}\rho_A^{r_1} \mathrm{Tr}\rho_A^{r_2} \rangle$$

exact result:

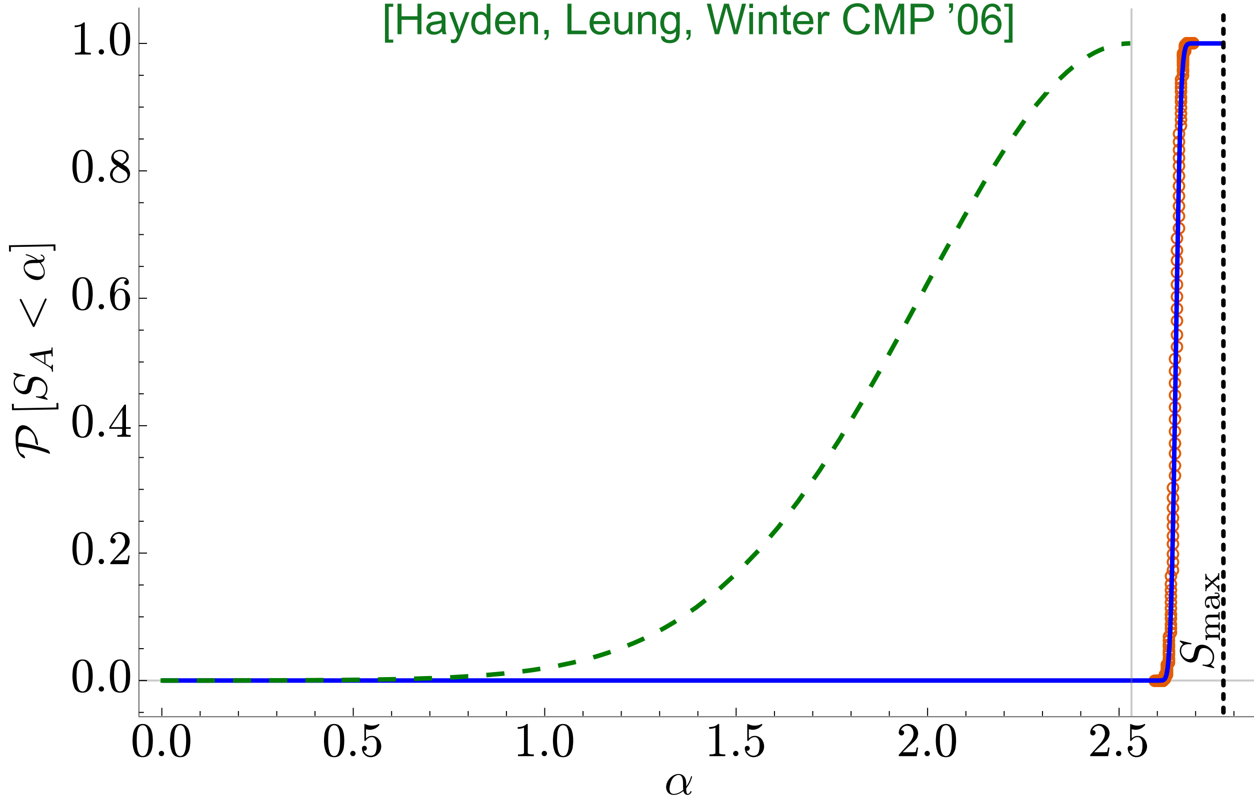
$$\Delta S_A^2 = \frac{d_A + d_B}{d_A d_B + 1} \Psi'(d_B + 1) - \Psi'(d_A d_B + 1) - \frac{(d_A - 1)(d_A + 2d_B - 1)}{4 d_B^2 (d_A d_B + 1)}$$

asymptotic:

$$\Delta S_A \approx \frac{1}{d_A d_B} \sqrt{\frac{d_A^2 - 1}{2}} \qquad d_B \gg 1 \qquad \Delta S_A \ll 1$$

The average entropy is typical! $\qquad d_A \leq d_B$

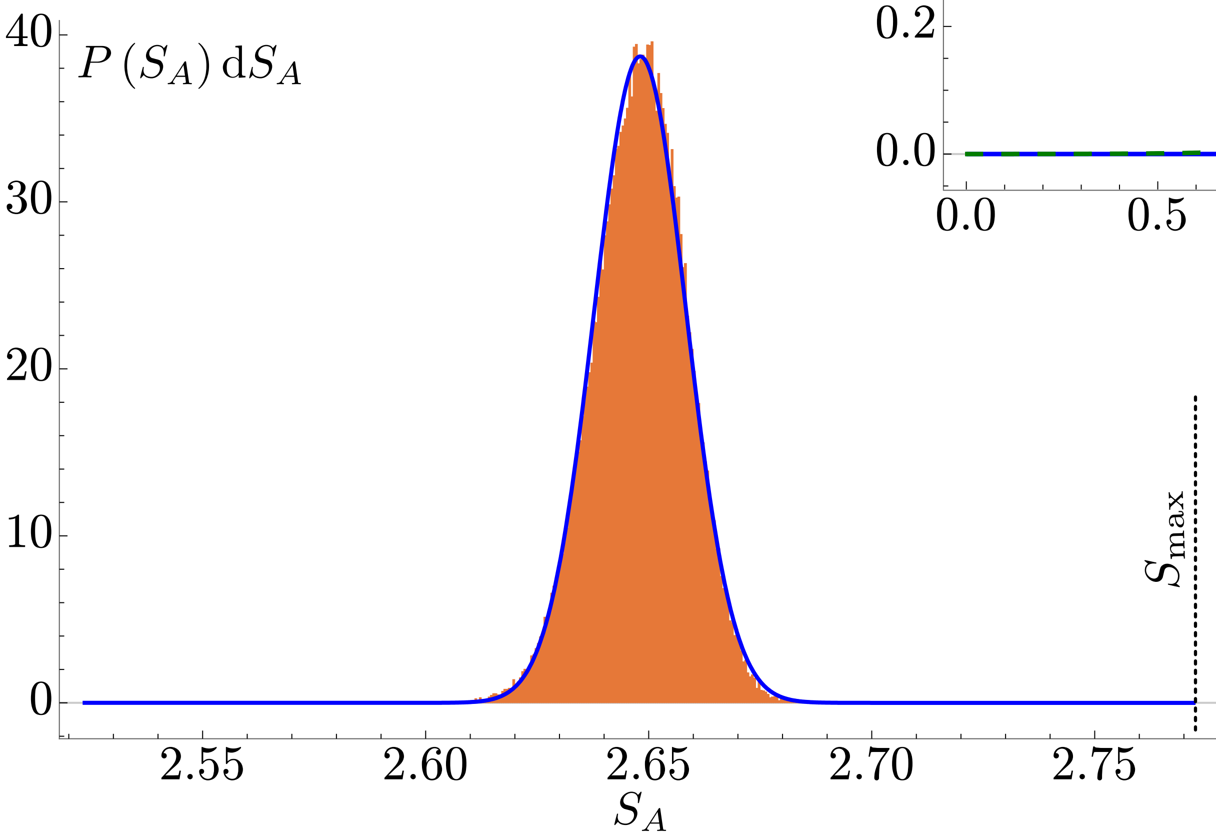
$10^5$ samples 200 bins $N_A = 4$ $N_B = 6$




Concentration of measure bounds

$$\mathcal{P}\left[S_A < \alpha\right] \leq e^{-\frac{(\alpha-\mu_b)^2}{2\sigma_b^2}}$$

$$\mu_b = \log d_A - \frac{d_A}{2d_B} \quad \sigma_b = \frac{2\pi \log d_A}{\sqrt{d_A d_B - 1}}$$




Hausdorff's moment problem

Any moment is computable (skewness)

Normal distribution is a good approximation

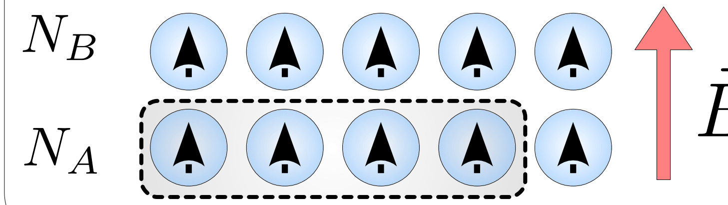
[Bianchi, P.D. PRD 2019]

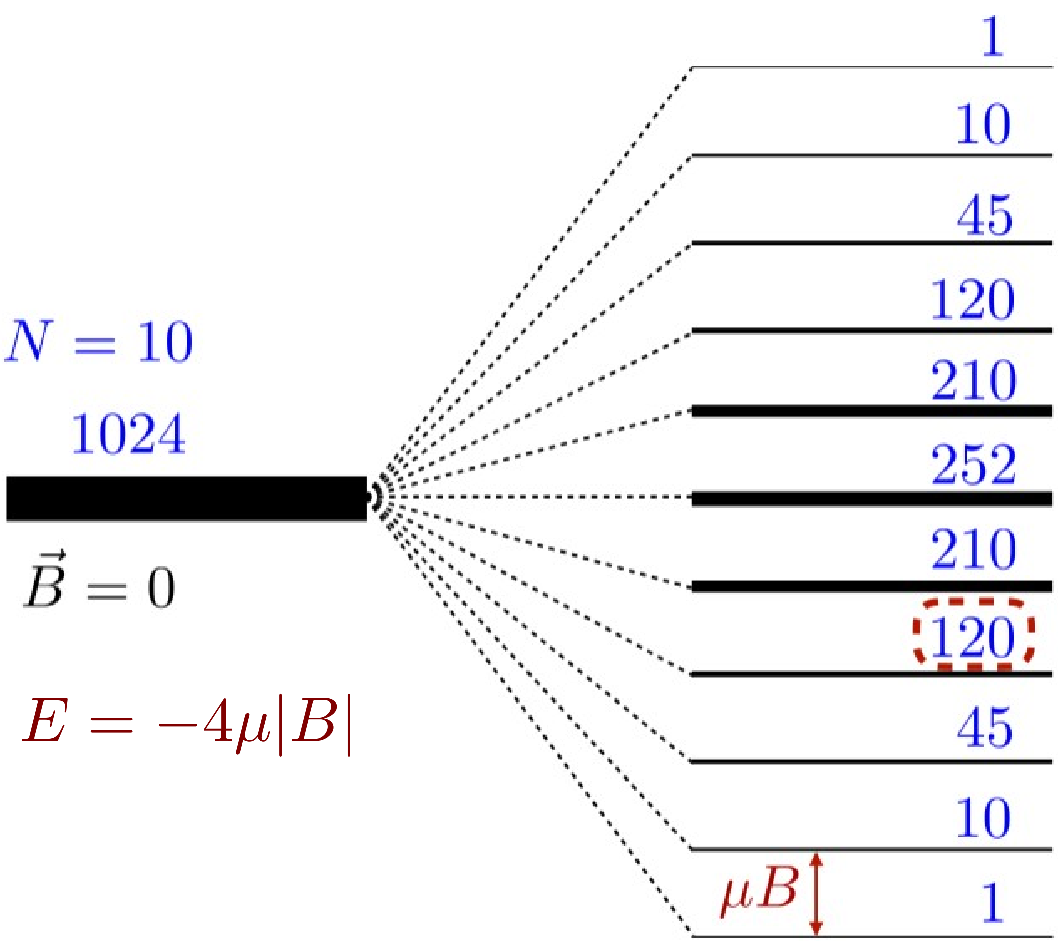
# Typical entropy of a subsystem with a center (example)

Hamiltonian $H = \sum_i \mu \vec{\sigma}_i \cdot \vec{B}$

eigenspace with given energy (direct sum) $\boxed{\mathcal{H}^E} = \boxed{\bigoplus_\epsilon} \left( \mathcal{H}_A^{(\epsilon)} \bigotimes \mathcal{H}_B^{(\epsilon)} \right)$

N spins + magn. field




Paramagnetic ionic salt

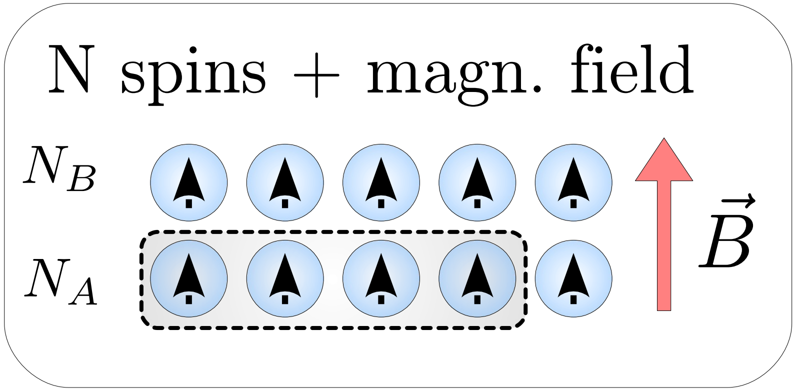
$\underline{Cu}SO_4.K_2SO_4.6H_2O$

$\mu \simeq 0.9 \times 10^{-23}$ J/T

$\simeq 0.6 \times 10^{-4}$ eV/T

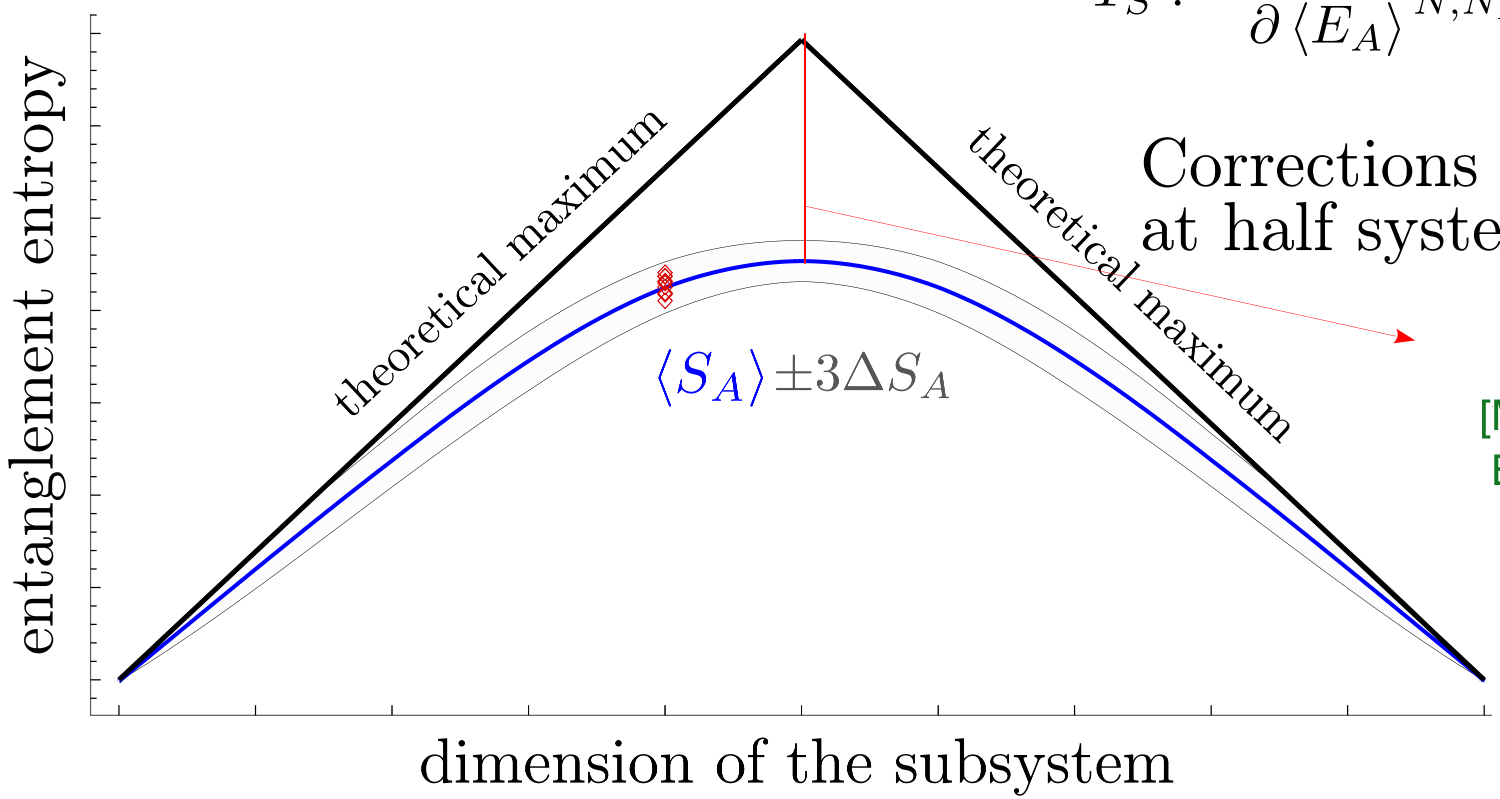
# Typical entropy of a subsystem with a center (Example)

$$\langle S_A \rangle = -\frac{N_A}{2}\left(1-\frac{E}{\mu B N}\right)\log\left(\frac{1-E/(\mu B N)}{2}\right) - \frac{N_A}{2}\left(1+\frac{E}{\mu B N}\right)\log\left(\frac{1+E/(\mu B N)}{2}\right)$$




The average entropy is typical

[Bianchi, P.D. PRD 2019]




Temperature from entanglement

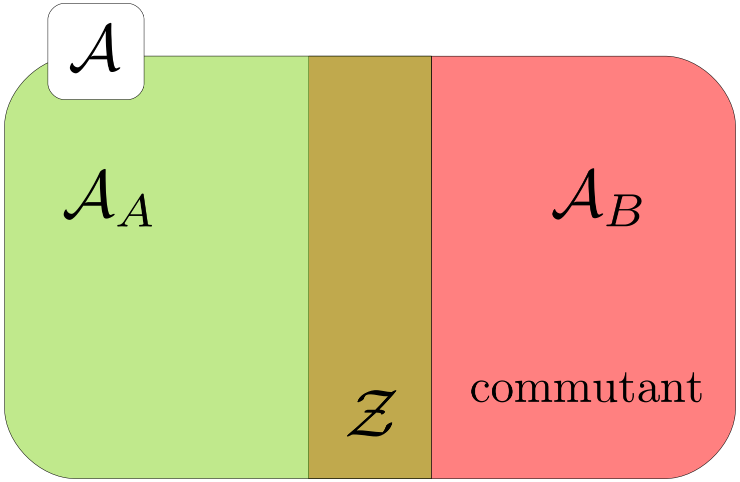
$$T_S := \frac{\partial \langle S_A \rangle}{\partial \langle E_A \rangle}_{N,N_A} = \frac{\mu B}{\operatorname{arctanh} E/(\mu B N)}$$

Corrections to entanglement at half system (heat capacity)

$$\Delta S_{half} = \pi\sqrt{N}\sqrt{C}$$

[Murthy, Srednicki PRE 2019, Bianchi, P.D. wip]

# Typical entropy in presence of a center [Bianchi, P.D. PRD 2019]




Hilbert space structure:

$$\mathcal{H} = \bigoplus_{\zeta \in \mathcal{Z}} \left( \mathcal{H}_A^{(\zeta)} \bigotimes \mathcal{H}_B^{(\zeta)} \right)$$

$$|\psi\rangle = \sum_\zeta \sqrt{p_\zeta}\, |\phi_A^{(\zeta)}\rangle |\phi_B^{(\zeta)}\rangle$$

Entanglement entropy

$$S_A(\psi) = \sum_\zeta p_\zeta S_A(|\phi_A^{(\zeta)}\rangle |\phi_B^{(\zeta)}\rangle) - \sum_\zeta p_\zeta \log p_\zeta$$

The average entropy is typical

$$\langle S_A(\psi)\rangle = \sum_\zeta \frac{d_{A\zeta} d_{B\zeta}}{d} \langle S_{A\zeta}\rangle + \Psi(d+1) - \Psi(d_{A\zeta} d_{B\zeta} + 1)$$

Exact formula. Variance and other momenta also computed.

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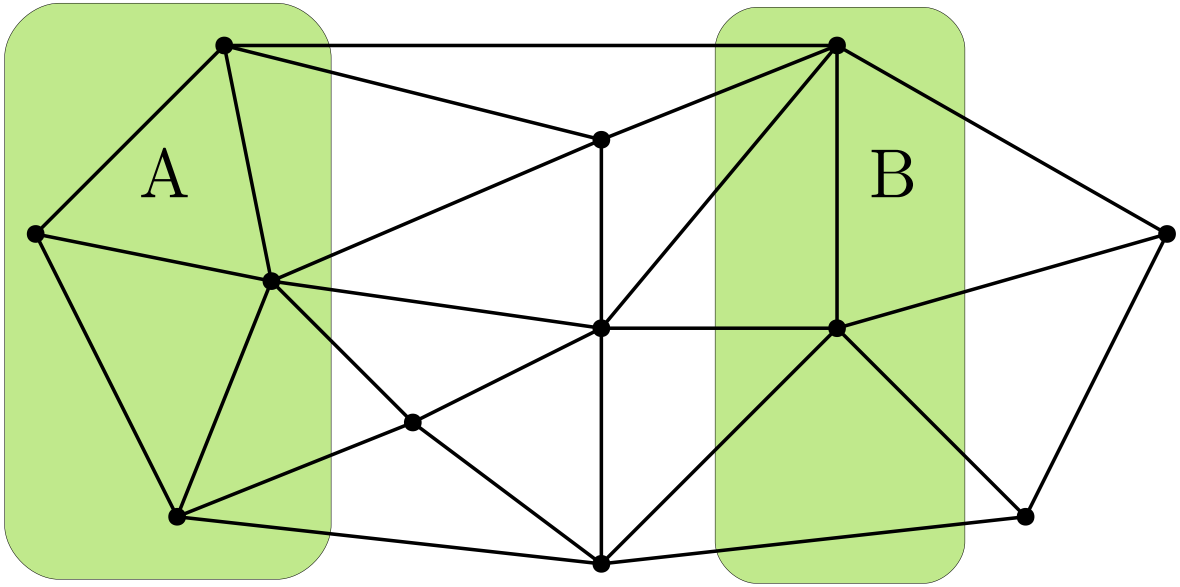
# Information-theoretic bounds on correlations




State:

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Bounded by mutual information

$$\frac{1}{2}\left(\frac{\mathcal{C}\left(O_A . O_B\right)}{\|O_A\|\|O_B\|}\right)^2 \leq S_A(\psi) + S_B(\psi) - S_{AB}(\psi)$$

[Wolf, Verstraete, Hastings, Cirac PRL 2008]

random states (large dimension) follow a volume law
= no correlations

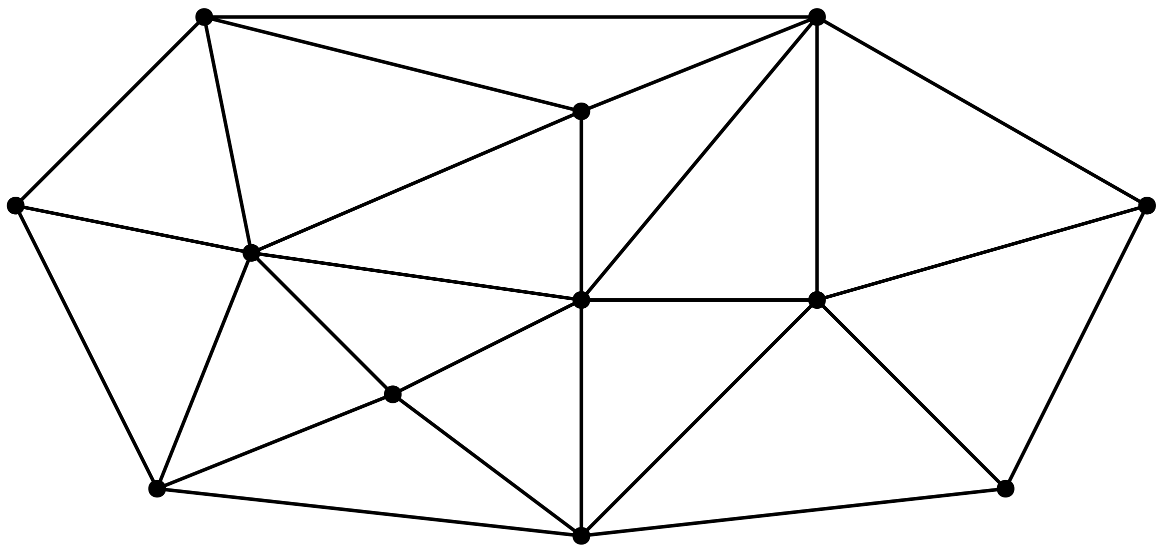
# Bell-network states

$$\mathcal{H}_\Gamma = \bigoplus_{j_\ell} \bigotimes_n \mathcal{H}_n$$

## Gluing quantum polyhedra with entanglement

[Bianchi, Baytas, Yokomizo, PRD 2018]

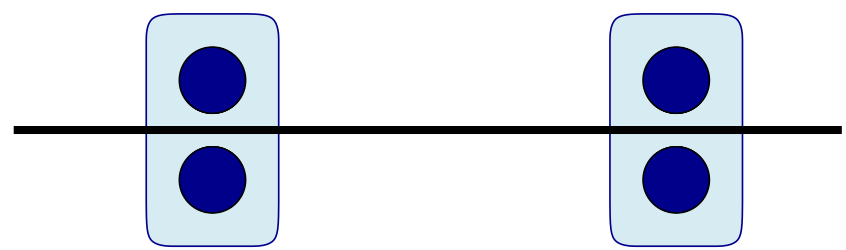
fluctuations of nearby quantum polyhedra are in general uncorrelated (twisted geometry)



## Use squeezed vacua techniques to correlate shapes link by link

[Bianchi, Hackl, Guglielmon, Yokomizo, PRD 2016]

## Enlarge Hilbert space, squeeze the oscillators and project into $\mathcal{H}_\Gamma$



Schwinger rep of SU(2)

$$|\Gamma, \lambda_\ell, \mathcal{B}\rangle = \sum_{j_\ell} \prod_\ell \left(1 - |\lambda_\ell|^2\right) \lambda_\ell^{2j_\ell} \sqrt{2j_\ell + 1}\, |\Gamma, j_\ell, \mathcal{B}\rangle \qquad |\Gamma, j_\ell, \mathcal{B}\rangle = \sum_{i_n} \mathcal{S}_\Gamma(j_\ell, i_n) \bigotimes_n |i_n\rangle$$

$\lambda_\ell \in \mathbb{C}$ link squeezing parameter (average area, extrinsic curvature)

symbol of the graph

# Bell-network states (analytic and numerics)

$$\mathcal{H}_\Gamma = \bigoplus_{j_\ell} \boxed{\bigotimes_n \mathcal{H}_n}$$

$$|\Gamma, j_\ell, \mathcal{B}\rangle = \sum_{i_n} \mathcal{S}_\Gamma(j_\ell, i_n) \bigotimes_n |i_n\rangle$$

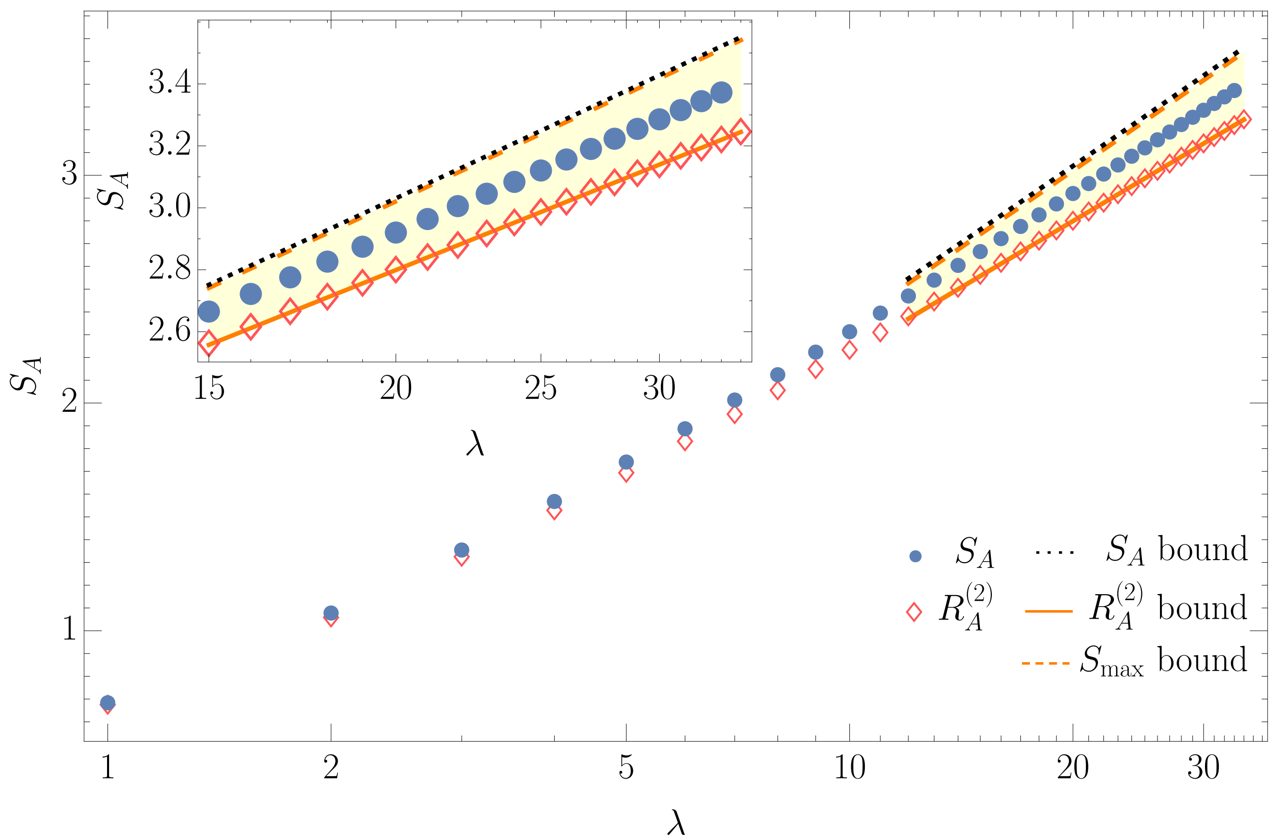
[Bianchi, P.D, Vilensky PRD 2019]

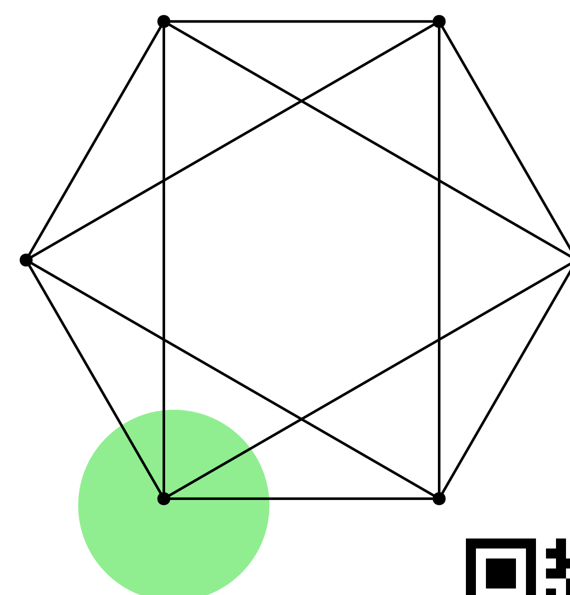
## Asymptotic bound on the entanglement entropy

$$j_\ell \to \lambda j_\ell$$

$$\left(|\partial A| - \frac{C_{\Gamma,A}}{2}\right) \log\lambda \le S_A \le \left(|\partial A| - \frac{3}{2}\right) \log\lambda$$

Numeric computations are available, at the moment, only for small graphs (but large spins)



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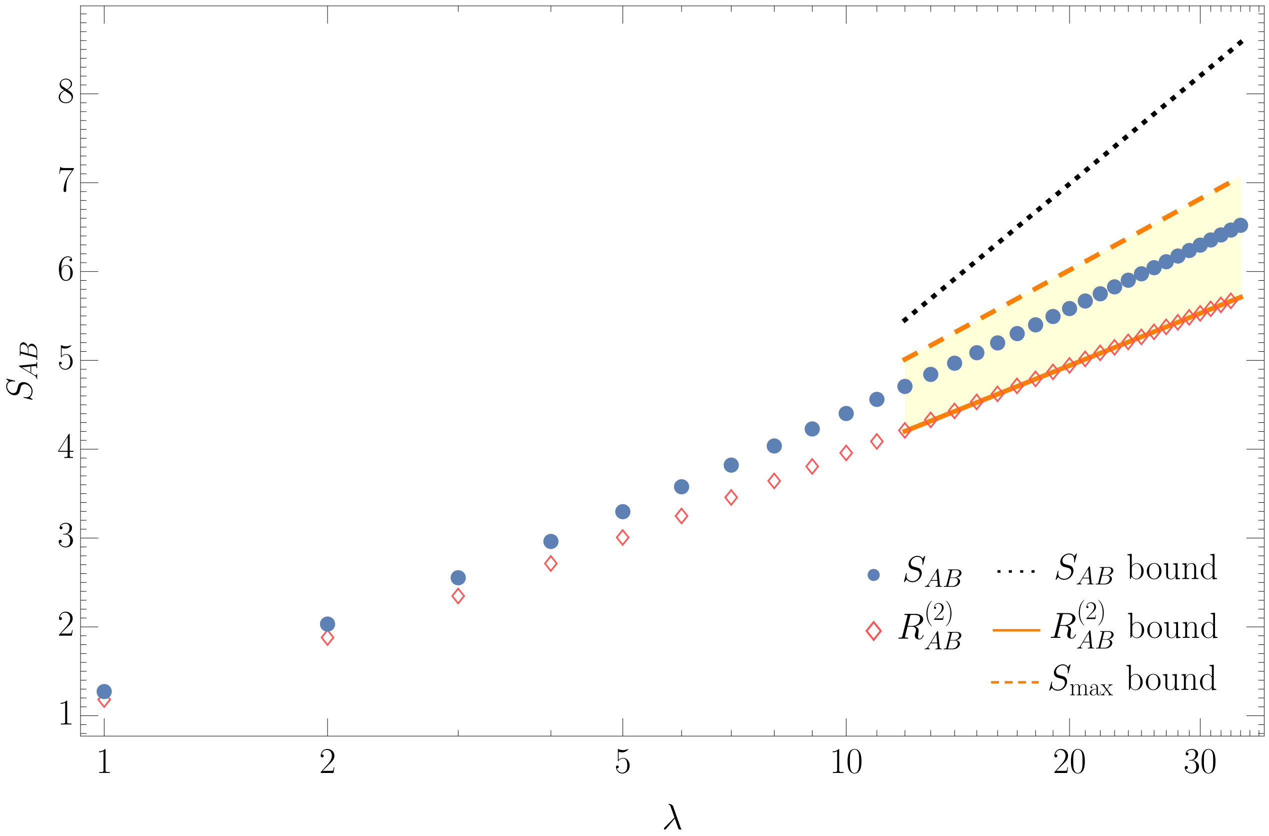


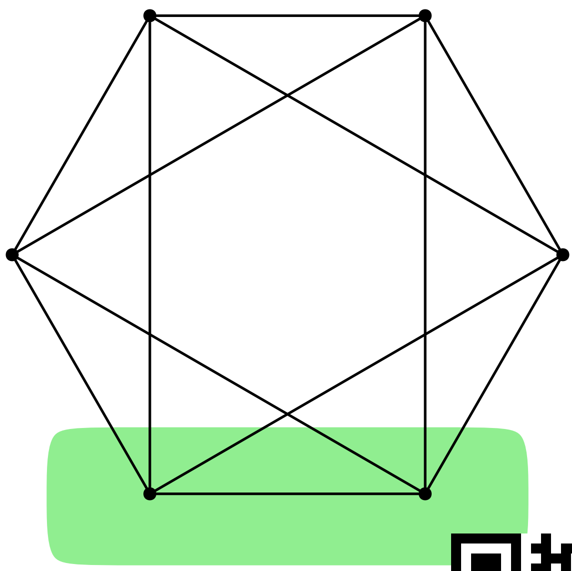

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Numeric computations are available, at the moment, only for small graphs (but large spins)

From numerical fit we can infer the bound on correlations

$$\frac{1}{2}\left(\frac{\mathcal{C}\left(O_A.O_B\right)}{\|O_A\|\|O_B\|}\right)^2 \leq 0.06 \log\lambda$$

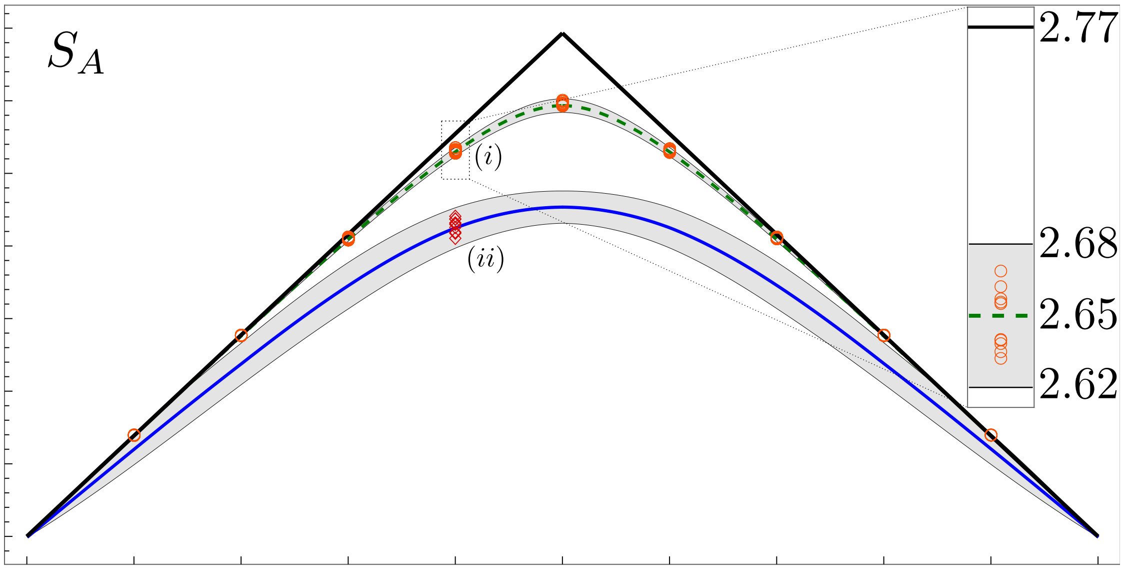
while for random states we obtain

$$\frac{1}{2}\left(\frac{\mathcal{C}\left(O_A.O_B\right)}{\|O_A\|\|O_B\|}\right)^2 \leq O(\lambda^{-1})$$

# Conclusions

## Page curve and its variance:

- Random states have typical entropy
- Unlikely to have maximum entropy
- Concentration of measure
- Vanishing correlations
- Even in presence of a center
- Temperature from entanglement
- Half volume correction (heat capacity)




## Entanglement entropy in a Bell-network state:

- Analytic asymptotics
- Area-law from intertwiner entanglement
- Numerical code
- Non-vanishing intertwiner correlations