Pietro Donà

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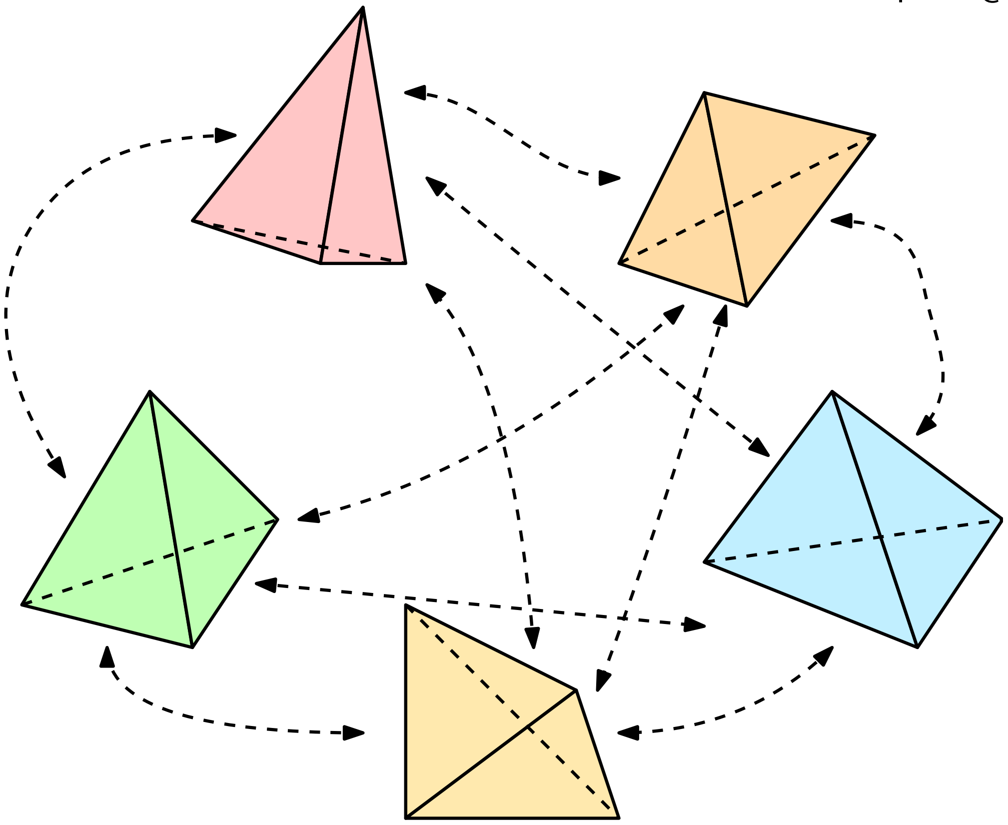

The role of local flatness in spin foam theories

UWO – 4th October 2022

Disclaimer

The topic is technical. I tried to focus on the general ideas as much as I could.

The interpretation is still work in progress.

For the students unfamiliar with spin foams:
Ignore the *how.* Try to grab the why!

EPRL spin foam theory

Engle-Pereira-Rovelli-Livine

- Dynamics for LQG as a path integral
- Regularized on simplicial triangulations
  (various extensions exists)
- Connection with discrete GR
- Numerical calculations are possible

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- Analytic calculations very complicated
- Where are Einstein equations?

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How? Why?

Regge geometries emerge in the large quantum numbers of many spin foam models

Bivector reconstruction theorem

John Barrett's results (2010ish)

Amazing mathematical result. However...

- Not constructive (proof of existence)
- Mixes a lot of ingredients (hard to follow)
- Vertex amplitude specific (awkward extensions)
- Slight changes requires a complete rework (i.e. Muxin & co. 10 years ago)
- It is just old

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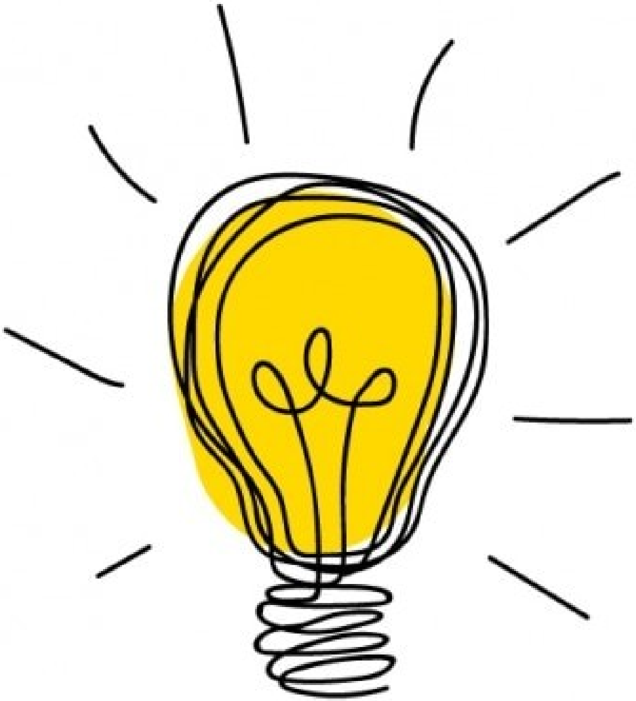
Different reasoning, similar conclusion, physically different models

(Topological BF SU(2), EPRL Lorentzian and Euclidean, EPRL extensions, Barrett-Crane)

Geometry appears always in same way! Why?

Idea!



The emergence of Regge geometries
does not depend on the details
of the spin foam model.

The key ingredient is

Local flatness

An overlooked property...

A short math interlude

Spinors are not complicated!
They are an extremely powerful tool for calculations.

$$|z\rangle := \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} \in \mathbb{C}^2$$

Using indices and tensors is more elegant, but I always mess up. I prefer Dirac's notation.

Complex structure $\langle z| := (\bar{z}_0,\ \bar{z}_1)$

Scalar product $\langle w|z\rangle = \bar{w}_0 z_0 + \bar{w}_1 z_1$

Dual spinor $|z] := \begin{pmatrix} -\bar{z}_1 \\ \bar{z}_0 \end{pmatrix}$

A spinor and its dual $|z\rangle, |z]$ are linearly independent, orthogonal, have opposite chirality, they form a basis of the spinoral space, and have a very interesting geometrical interpretation

Geometrical interpretation

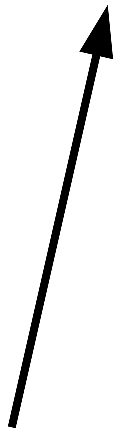
To simplify formulas I will work with norm 1 spinors $\langle z|z\rangle = 1$

Not a restriction. I am just lazy!

With each (unit) spinor we can build a (unit) vector

$$\langle z|\vec{\sigma}|z\rangle = -\vec{n}$$

independent from the phase of the spinor.



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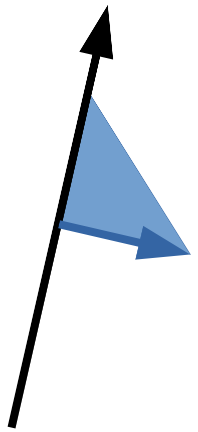


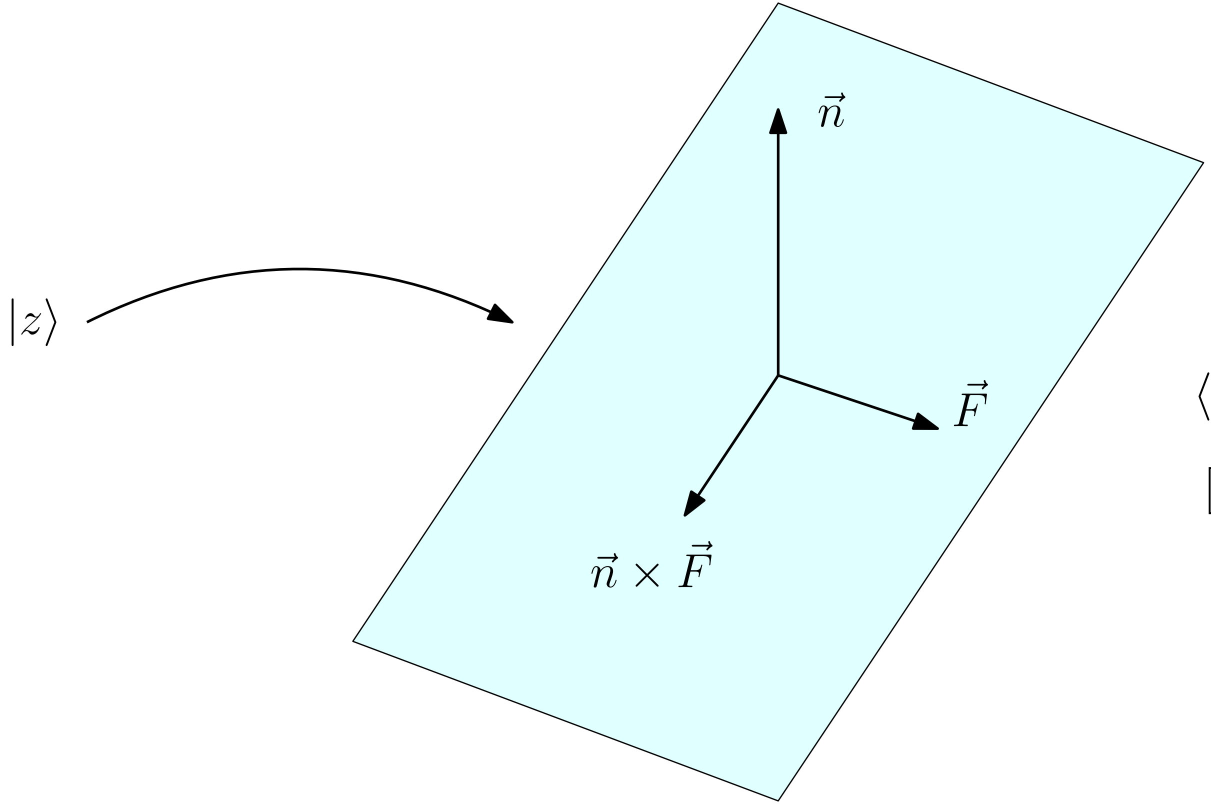
There is more (flag/phase info)

(unit) frame vector $[z|\vec{\sigma}|z\rangle = i\vec{F} + \vec{n} \times \vec{F}$

orthogonal to the vector

$\left\{\vec{n}, \vec{F}, \vec{n} \times \vec{F}\right\}$ orthonormal basis of $\mathbb{R}^3$ (framed plane)

Spinors as framed planes



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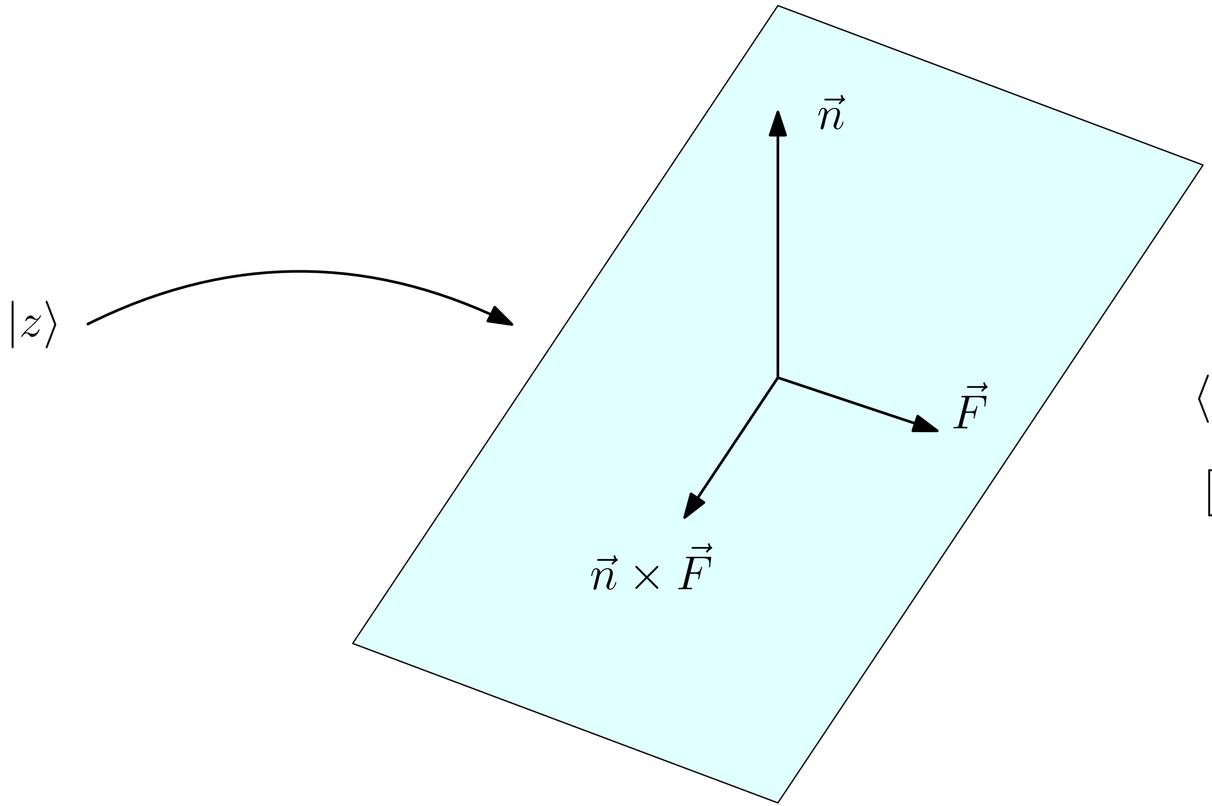

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Déjà vu?

It is the same geometrical picture of LQG

(twistorial phase space, twisted geometries - Simone, Etera, Laurent …)

Why spinors as framed planes?

Direct interpretation in terms of LQG variables.

No *natural action* of Lorentz group in $\mathbb{R}^3$ you need to define the embedding in a larger space.
(model dependent, depends on the choice of representation)

There is a *natural action* of Lorentz group on spinors! (technically one for each chirality)

What can I do without defining an embedding? (model independent)

Holonomies as map between framed planes

Clever parametrization of $SL(2,\mathbb{C})$

$$g = e^{\frac{\omega}{2}}|w\rangle\langle z| + e^{-\frac{\omega}{2}}|w][z|$$

A source and a target spinors and a complex number (redundant, two extra phases, helps with interpretation)

Adapted coordinates to the source and target space. The matrix is diagonal

$$\begin{pmatrix} e^{\frac{\omega}{2}} & 0 \\ 0 & e^{-\frac{\omega}{2}} \end{pmatrix}$$

$$a|z\rangle + b|z] \longrightarrow \begin{pmatrix} a, \\ b \end{pmatrix}$$

$$c|w\rangle + d|w] \longrightarrow \begin{pmatrix} c, \\ d \end{pmatrix}$$

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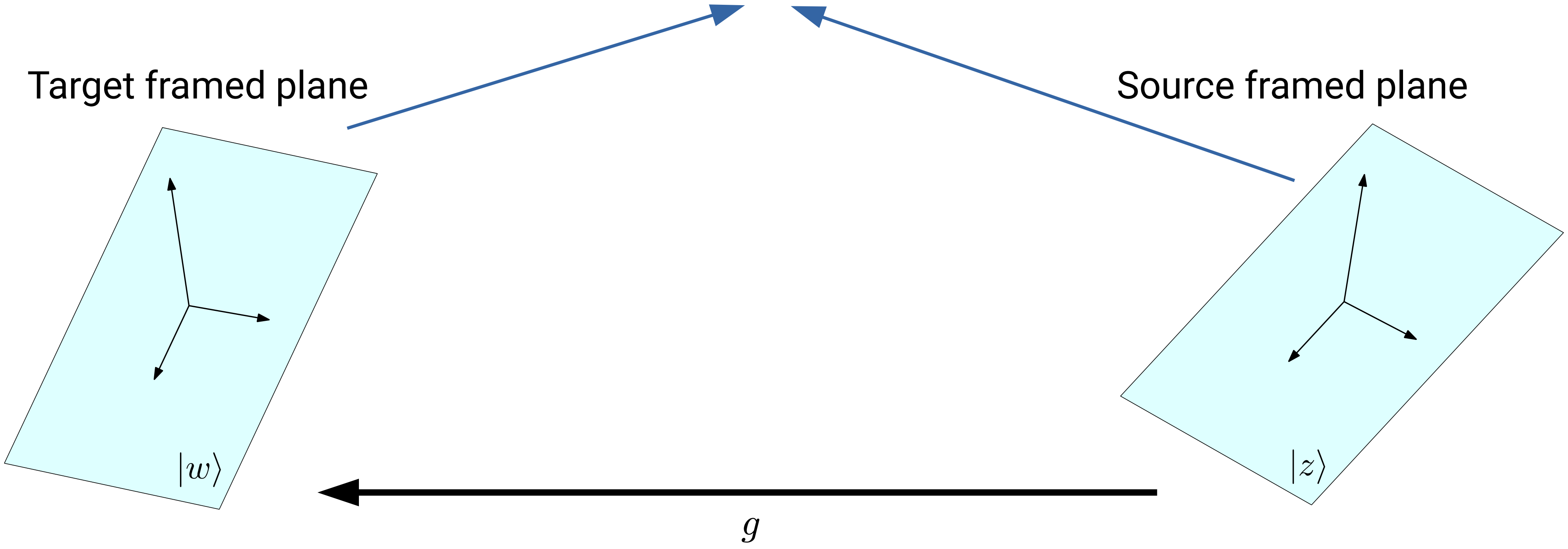

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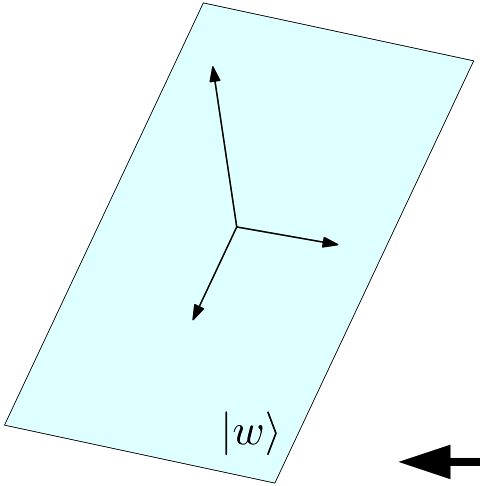
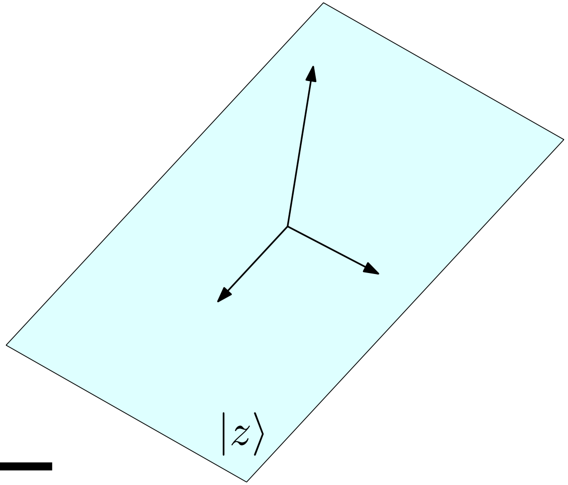
Target framed plane



Source framed plane



$\mathrm{Re}\ \omega$ Represent a boost between framed planes (as def)

$\mathrm{Im}\ \omega$ Represent a twist of the frames (set $\mathrm{Re} = 0$)

g

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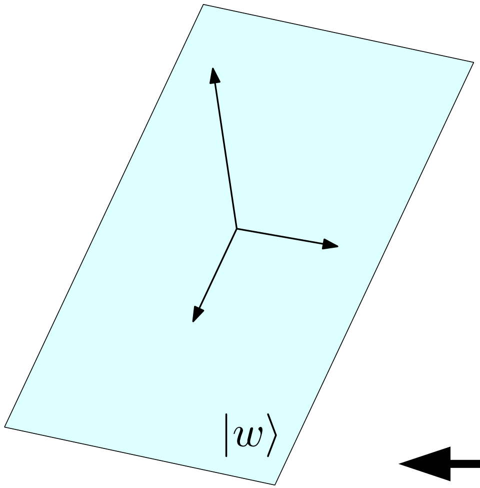

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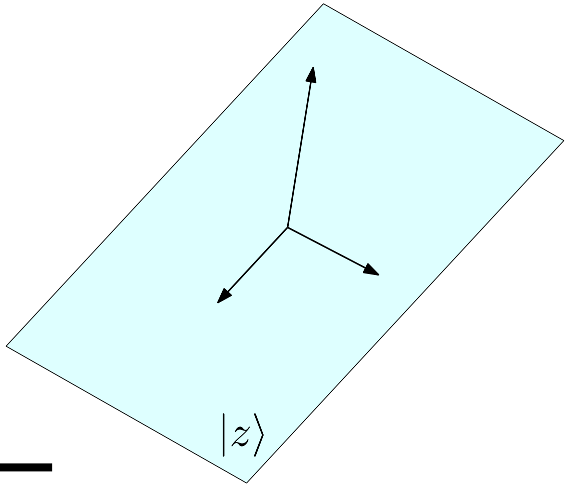

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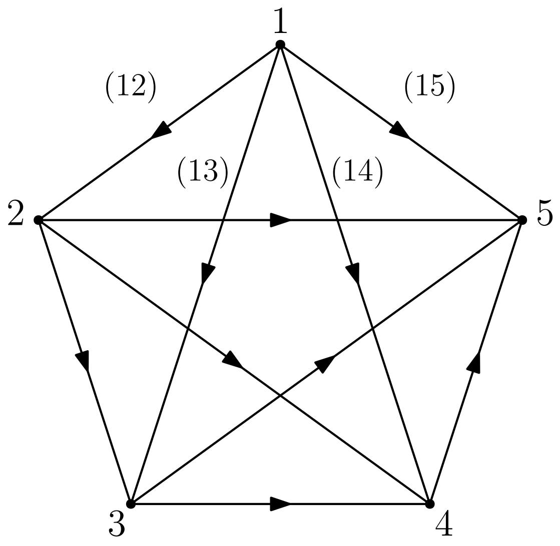
Geometry intrinsically associated to the holonomy

(it is the "same" as in twisted geometries)

Setting the scene for local flatness

A general 2-complex is made of multiple 4-simplices. (No geometric data, just combinatorial information)

Consider the 2-complex of one 4-simplex.



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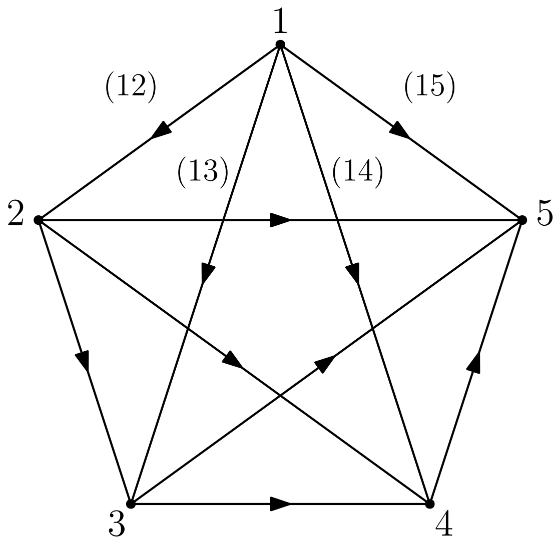

Five edges (dual to tetrahedra)

a=1, 2, 3, 4, 5

Ten faces (wedges) given by a couple of edges

(ab)=(12), (13), (14), (15), …

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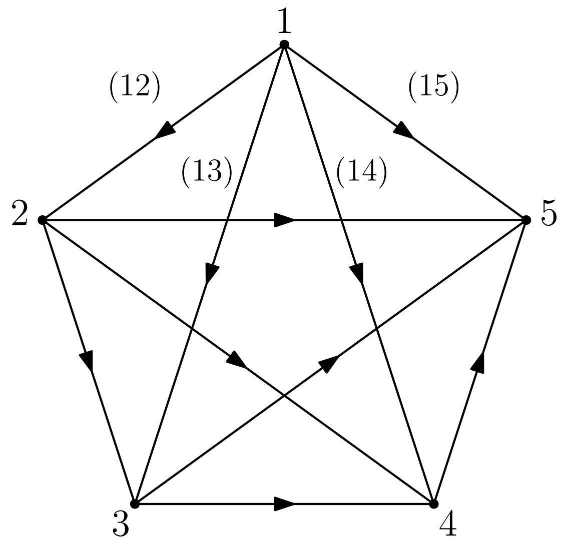

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Associate one $SL(2,\mathbb{C})$ holonomy to each wedge

$$g_{ab} = e^{\frac{\omega_{ab}}{2}} |z_{ba}] \langle z_{ab}| - e^{-\frac{\omega_{ab}}{2}} |z_{ba}\rangle [z_{ab}|$$

(changed convention slightly, helps with interpretation)

Orientation is conventional (a is the source and b is the target)

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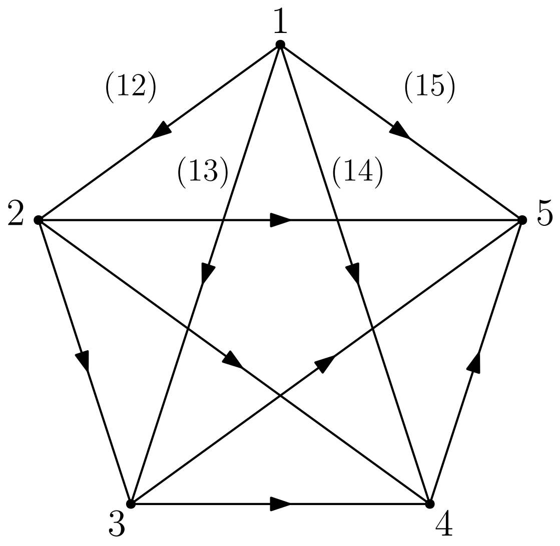

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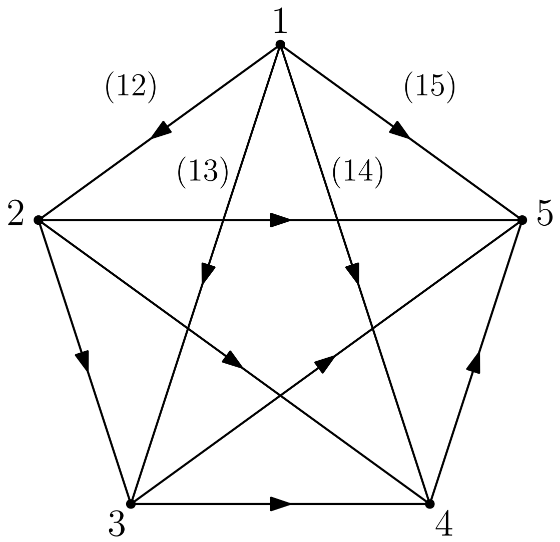
Ten holonomies describe the parallel transport from one edge to another. They associate four framed planes to each edge and a complex number to each face.

Holonomies and local flatness

Flat building blocks!



We require each 4-simplex to be flat! The 2-complex is locally flat.

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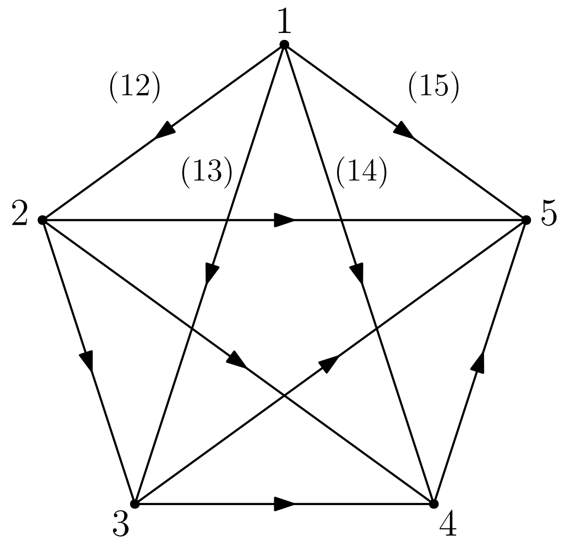

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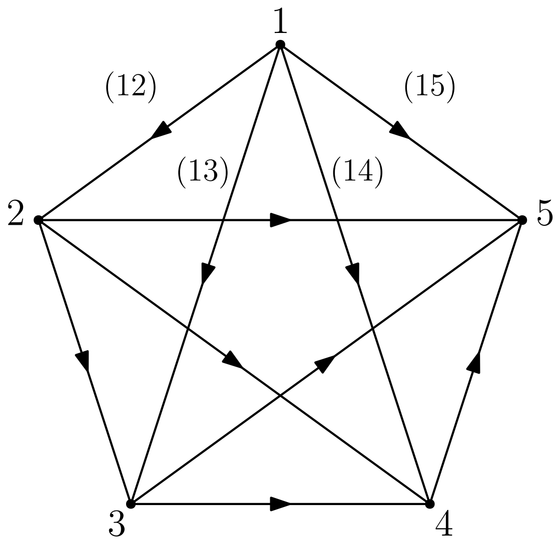

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Constraints on holonomies = constraints on the geometries (framed planes, spinors and complex angles)

How to solve? Smart projection on components

$$[z_{ac}| g_{ab}^{-1} | z_{bc}\rangle = [z_{ac}| g_{ca} g_{bc} | z_{bc}\rangle$$
$$\langle z_{ac}| g_{ab}^{-1} | z_{bc}] = \langle z_{ac}| g_{ca} g_{bc} | z_{bc}]$$

Plus other two. 4 complex scalar equations.
Combine to find...

Glossary

3d dihedral angles
(angle between 2 framed planes at the same edge)

$$\cos\phi^a_{bc} = \vec{n}_{ab}\cdot\vec{n}_{ac} = 2|\langle z_{ab}|z_{ac}\rangle|^2 - 1$$

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Spherical cosine law and sine law

(local embedding of 3D hyperplanes in 4D – signature?)

$$\cos\hat{\theta}^c_{ab} = \frac{-|\langle z_{ca}|z_{cb}]|^2 + |\langle z_{ab}|z_{ac}\rangle|^2|\langle z_{ba}|z_{bc}\rangle|^2 + |\langle z_{ac}|z_{ab}]|^2|\langle z_{ba}|z_{bc}]|^2}{2|\langle z_{ac}|z_{ab}\rangle\langle z_{ac}|z_{ab}]\langle z_{ba}|z_{bc}\rangle\langle z_{ba}|z_{bc}]|} = \frac{\cos\phi^c_{ab} + \cos\phi^a_{cb}\cos\phi^b_{ac}}{\sin\phi^a_{cb}\sin\phi^b_{ac}}$$

$$\sin\phi^b_{ac}\sinh\hat{\theta}^c_{ab} = \sin\phi^c_{ab}\sinh\hat{\theta}^b_{ac}$$

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Twist angle
(measure twist between frames using a third as reference -
the same one defined by Bianca and Jimmy or Simone and Fabio)

$$\xi^c_{ab} = \arg\left(\frac{\langle z_{ac}|z_{ab}]\langle z_{ab}|z_{ac}\rangle}{\langle z_{bc}|z_{ba}\rangle[z_{ba}|z_{bc}\rangle}\right)$$

Holonomies and local flatness

Local flatness

$$g_{ca}g_{bc}g_{ab} = \mathbb{1}$$

$=$

Complex angle determined by spinors

$$\cosh(\omega_{ab} + i\xi^c_{ab}) = \cos\hat{\theta}^c_{ab}$$

$$\sin\phi^b_{ac}\sinh(\omega_{ab} + i\xi^c_{ab}) = \sin\phi^c_{ab}\sinh(\omega_{ca} + i\xi^b_{ac})$$

for every cycle = constrains also the spinors!

Solutions?

Studied by Me and Simone 2 years ago

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Solutions?

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Lorentzian sector $|\cos\hat{\theta}^c_{ab}| > 1$

The other sector is the topological one (vector and euclidean) with $\mathrm{SU}(2)$ holonomies

Lorentzian geometries (need edge independence = angle matching = "shape" matching)

$$\omega_{ab} = \epsilon\theta_{ab} + i\epsilon\chi_{ab}\pi - i\xi_{ab}$$

orientation, dihedral angle, local causal structure, twist angles

Holonomies and local flatness

Local flatness

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=

Complex angle determined by spinors

+

angle matching conditions (strongly)

(restriction to the Lorentzian sector)

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In all <u>locally flat</u> Lorentzian spin foam models angle matched Lorentzian geometries (Regge) emerge!

NOTE. General! No amplitude, embedding map, semiclassical regime or critical point eqs.

Local flatness in the EPRL model

(all the spin foam models are locally flat)

$$A_v = \int \prod_a \mathrm{d}g_a \delta(g_1) \prod_{ab} D^{(\gamma j_{ab}, j_{ab})}_{j_{ab}\zeta_{ba} j_{ab}\zeta_{ab}} \boxed{(g_b^{-1} g_a)}$$

(edge holonomies)

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(edge holonomies)

Local flatness is imposed strongly

$$A_v = \int \left(\prod_{ab} \mathrm{d}g_{ab} D^{(\gamma j_{ab}, j_{ab})}_{j_{ab}\zeta_{ba} j_{ab}\zeta_{ab}} (g_{ab}) \right) \mathcal{C}_{LF}(g_{ab}, \cdots, g_{cd})$$

(wedge holonomies)

$$+$$

$$\mathcal{C}_{LF}(g_{ab}, \cdots, g_{cd}) = \delta\left(g_{13}^{-1} g_{23} g_{12}\right) \delta\left(g_{14}^{-1} g_{24} g_{12}\right) \delta\left(g_{15}^{-1} g_{25} g_{12}\right) \cdot \delta\left(g_{14}^{-1} g_{34} g_{13}\right) \delta\left(g_{15}^{-1} g_{35} g_{13}\right) \delta\left(g_{15}^{-1} g_{45} g_{14}\right) .$$

Curiosity: the necessity of regularize the amplitude correspond to the necessity of consider fundamental cycles!
Different regularization choice = different fundamental cycles choice

What is the role of the critical point equations?

$$A_v = \int \left(\prod_{ab} \mathrm{d}g_{ab} D^{(\gamma j_{ab}, j_{ab})}_{j_{ab}\boxed{\zeta_{ba}}\, j_{ab}\boxed{\zeta_{ab}}}(g_{ab}) \right) \mathcal{C}_{LF}(g_{ab}, \cdots, g_{cd})$$

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Alignment equations → The spinors (framed planes) parametrizing the holonomies and the boundary ones are the same

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Saddle point → **Closure conditions** (boundary described as framed tetrahedra) + **Alignment equations** (holonomies spinors coincide with boundary ones)

Action at the critical point

$$i\lambda \sum_{ab} j_{ab} \left(\gamma \mathrm{Re}\, \omega_{ab} + \mathrm{Im}\, \omega_{ab} \right)$$

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focus to the Lorentzian sector

$$\omega_{ab} = \boxed{\epsilon\theta_{ab}} + \boxed{i\epsilon\chi_{ab}\pi} - \boxed{i\xi_{ab}}$$

4D dihedral angle between the framed tetrahedra (ab)

Local causal structure

Twist between the framed tetrahedra (ab)

Other stuffs

Many vertices?

Analysis on vertex amplitudes independently
(local flatness + closure + alignment)

$+$

Extra alignment equations
(framed tetrahedra shared by different vertices coincide!)

Summing over the spins

Constraining face holonomies?

Singular support of the face distribution

(mystic result by Hellmann and Kaminski)

Naive flatness problem arises when you combine
Local flatness + singular support + alignment

Topological BF: $\delta(g_f)$ $\longrightarrow$ $g_f = \mathbb{1}$

EPRL: $f_{EPRL}(g_f)$ $\longrightarrow$ $g_f = e^{\frac{\omega_f}{2}}|\zeta\rangle\langle\zeta| + e^{-\frac{\omega_f}{2}}|\zeta][\zeta|$ with $\gamma\mathrm{Re}\omega_f + \mathrm{Im}\omega_f = 0 \mod 4\pi$

Conclusions

Local flatness is responsible of the emergence of Regge geometry (vertex by vertex) in spin foam models

Any locally flat spin foam model knows about Lorentzian 4-simplices (topological sector has SU(2) holonomies)

Secondary simplicity constraints? (imposed strongly)

Quantum simplicity constraints (alignment + action)

Connection with effective spin foam models (Area-angle Regge calculus)

Separation of ingredients is key to innovate (maybe new model? Simpler to do calculations! Top down construction! I have no concrete proposal.)