Pietro Donà

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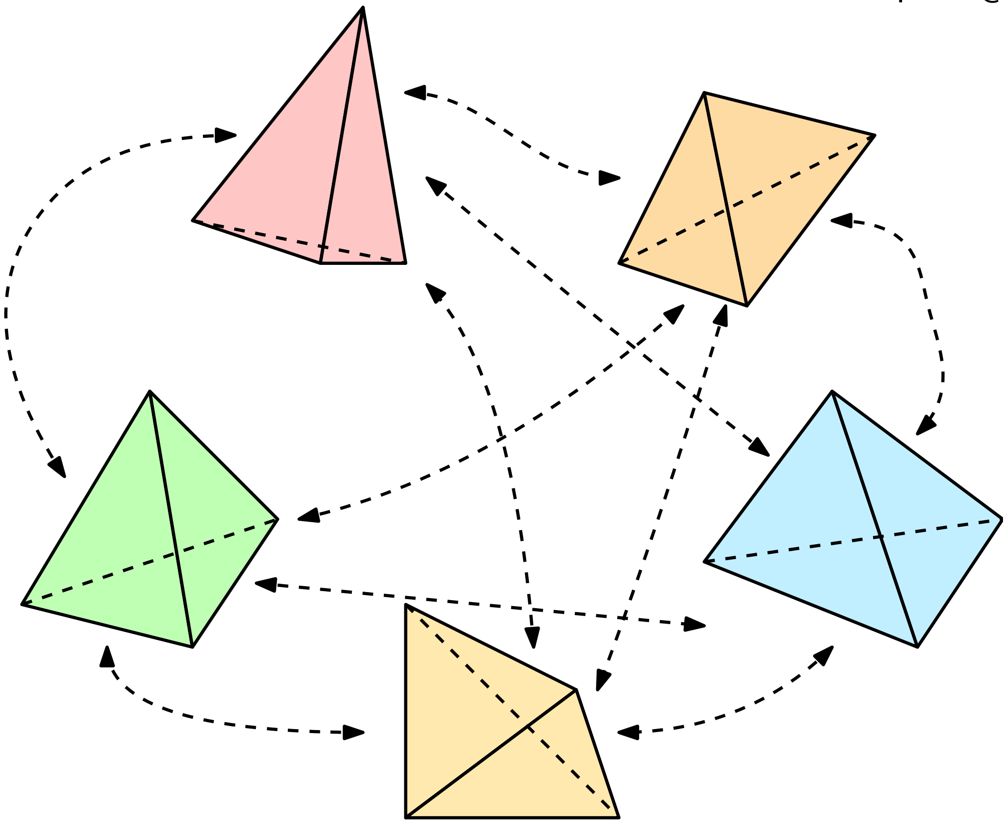

The role of local flatness in spin foam theories

Friedrich-Schiller University Jena – 7th November 2022

EPRL spin foam theory

- Dynamics for LQG as a path integral
- Regularized on simplicial triangulations
  (various extensions exists)
- Connection with discrete GR
- Numerical calculations are possible

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👍 How? 👎 Why?

Regge geometries emerge in the large quantum numbers of many spin foam models

Bivector reconstruction theorem

John Barrett's results (2010ish)

Amazing mathematical result. However...

- Not constructive (proof of existence)
- Mixes a lot of ingredients (hard to follow)
- Vertex amplitude specific (awkward extensions)
- Slight changes requires a complete rework (i.e. Muxin & co. 10 years ago)
- It is just old

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Different reasoning, similar conclusion, physically different models

(Topological BF SU(2), EPRL Lorentzian and Euclidean, EPRL extensions, Barrett-Crane)

Geometry appears always with the same mechanism! Why?

Idea!

The emergence of Regge geometries does not depend on the details of the spin foam model. It comes from

Local flatness

An overlooked property...

A short math interlude

Spinors are not complicated!
They are an extremely powerful tool for calculations.

$$|z\rangle := \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} \in \mathbb{C}^2$$

Using indices and tensors is more elegant, but I always mess up. I prefer Dirac's notation.

Complex structure $\langle z| := (\bar{z}_0,\ \bar{z}_1)$

Scalar product $\langle w|z\rangle = \bar{w}_0 z_0 + \bar{w}_1 z_1$

Dual spinor $|z] := \begin{pmatrix} -\bar{z}_1 \\ \bar{z}_0 \end{pmatrix}$

A spinor and its dual $|z\rangle, |z]$ are linearly independent, orthogonal, have opposite chirality, they form a basis of the spinoral space, and have a very interesting geometrical interpretation

Geometrical interpretation

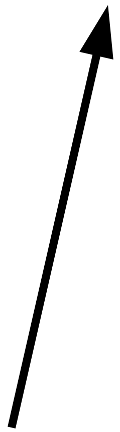
To simplify formulas I will work with norm 1 spinors $\langle z|z\rangle = 1$

Not a restriction. I am just lazy!

With each (unit) spinor we can build a (unit) vector

$$\langle z|\vec{\sigma}|z\rangle = -\vec{n}$$

independent from the phase of the spinor.



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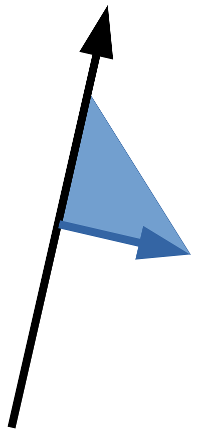


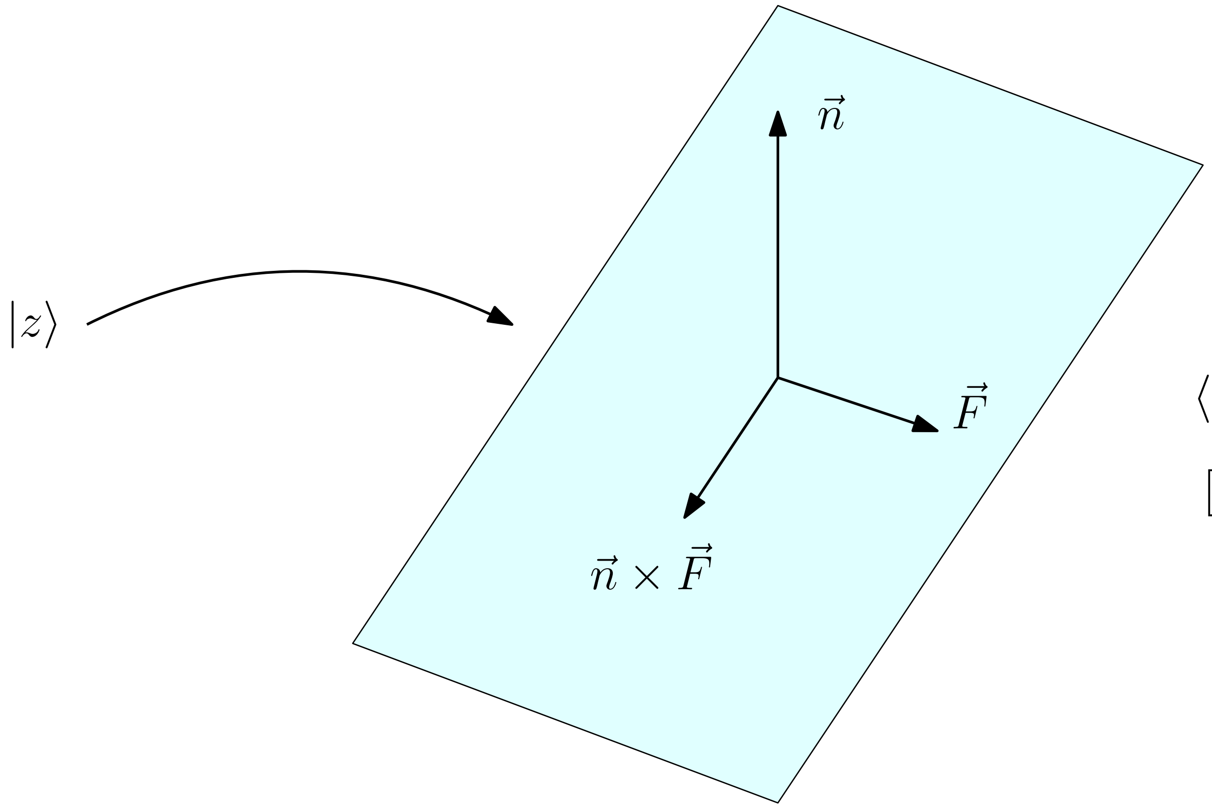
There is more (flag/phase info)

(unit) frame vector $[z|\vec{\sigma}|z\rangle = i\vec{F} + \vec{n} \times \vec{F}$

orthogonal to the vector

$\left\{\vec{n}, \vec{F}, \vec{n} \times \vec{F}\right\}$ orthonormal basis of Euclidean $\mathbb{R}^3$ (framed plane)

Spinors as framed planes

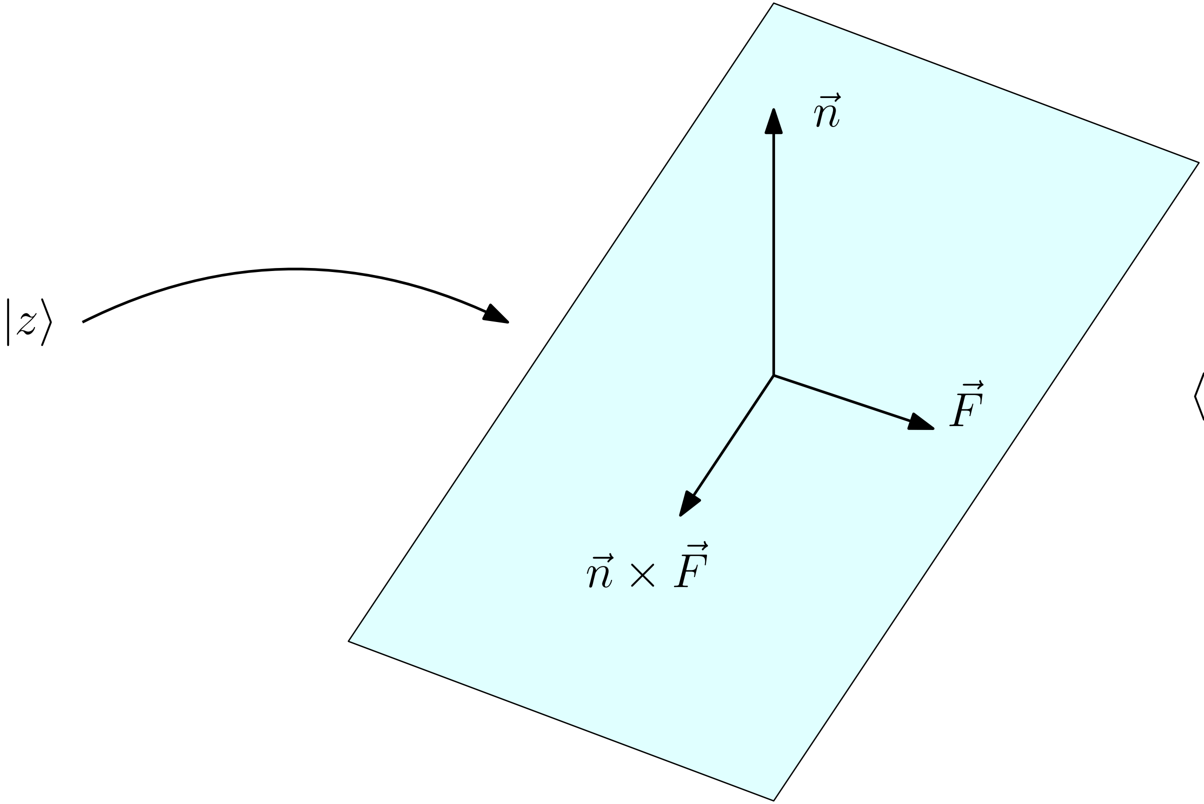


$$\langle z|\vec{\sigma}|z\rangle = -\vec{n}$$

$$[z|\vec{\sigma}|z\rangle = i\vec{F} + \vec{n} \times \vec{F}$$

Framed plane in Euclidean 3D space

Spinors as framed planes

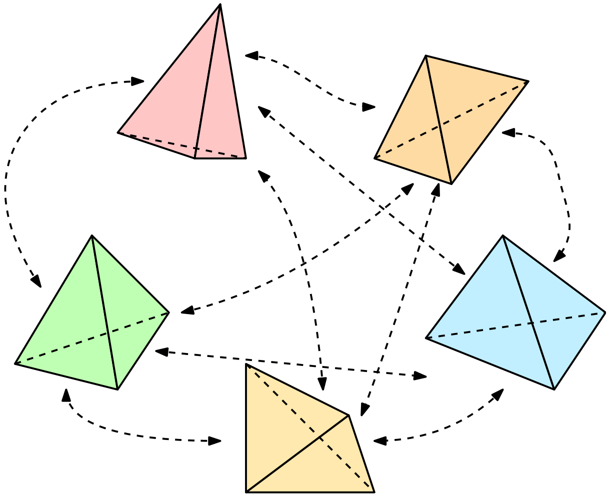


$$\langle z|\vec{\sigma}|z\rangle = -\vec{n}$$

$$[z|\vec{\sigma}|z\rangle = i\vec{F} + \vec{n} \times \vec{F}$$

Déjà vu?
It is the same geometrical picture of LQG
(twistorial phase space, twisted geometries - Simone, Etera, Laurent …)

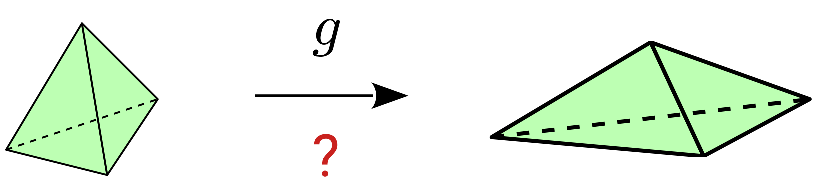
Why spinors as framed planes?



Direct interpretation in terms of LQG variables.
(twisted geometries are also parametrized in terms of spinors)

There is a *natural action* of Lorentz group on spinors!
(model independent, independs on the choice of representation, no extra ingredients)

$$|z\rangle \rightarrow g\,|z\rangle$$



(EPRL suggests electric part of gamma simple bivectors)

What can we infer from elementary properties?

Holonomies as map between framed planes

Clever parametrization of $SL(2,\mathbb{C})$

$$g = e^{\frac{\omega}{2}}|w\rangle\langle z| + e^{-\frac{\omega}{2}}|w][z|$$

A source spinor, a target spinor and a complex number (redundant, two extra phases, helps with interpretation)

Similar to twisted geometries parametrization of the Ashtekar holonomy. (particular adapted basis)

Geometry?

Holonomies as map between framed planes

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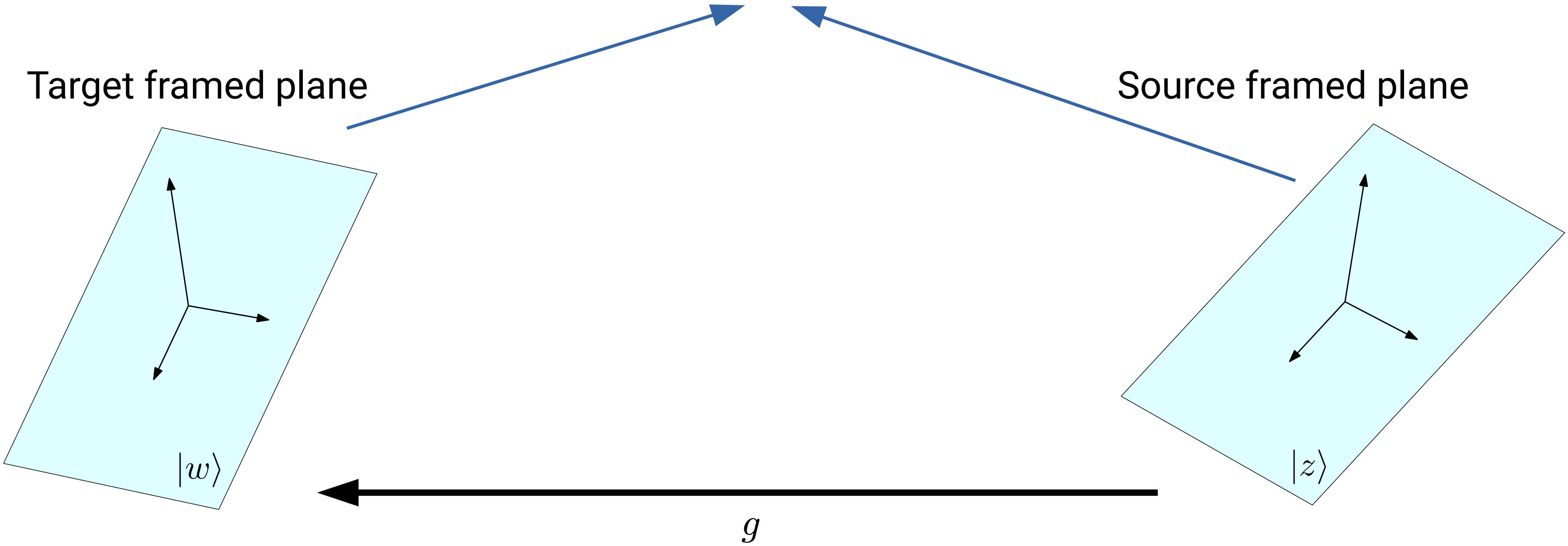

Holonomies as map between framed planes

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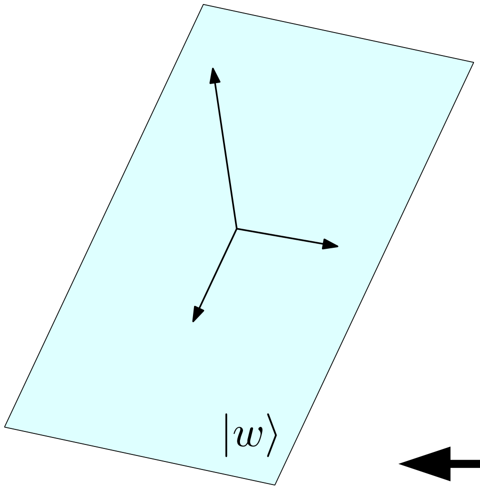
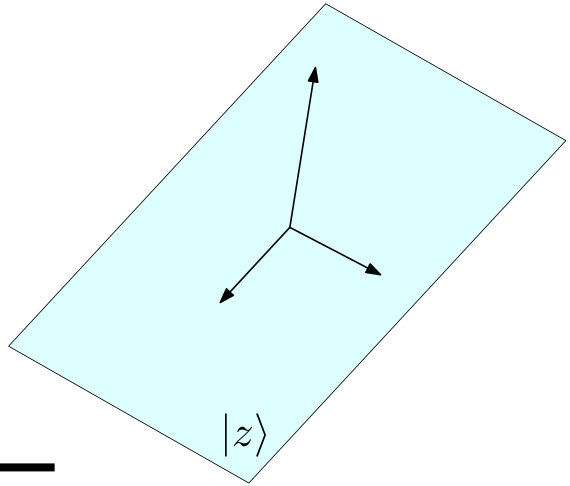
Target framed plane



Source framed plane



$\mathrm{Re}\ \omega$ Represent a boost between framed planes (polar)

$\mathrm{Im}\ \omega$ Represent a twist of the frames (misalignment)

g

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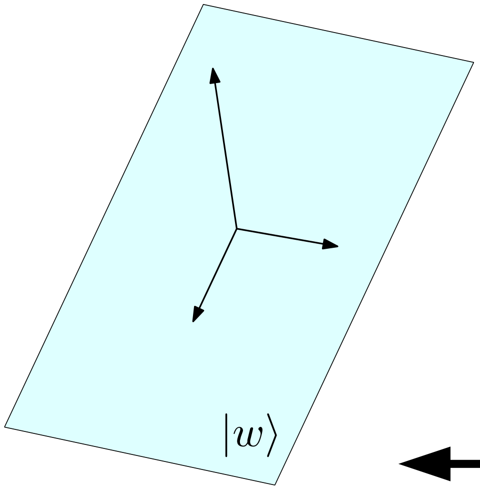

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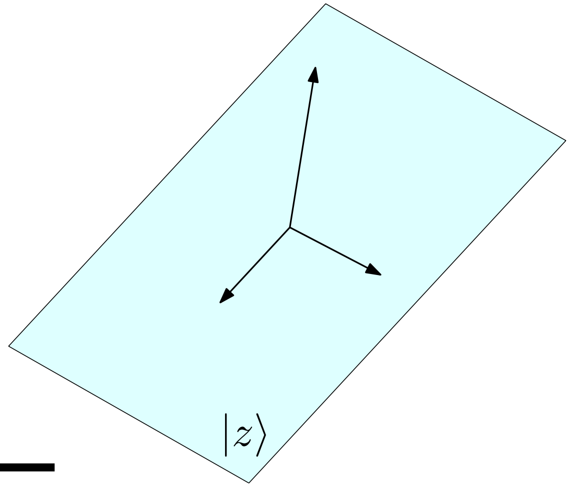

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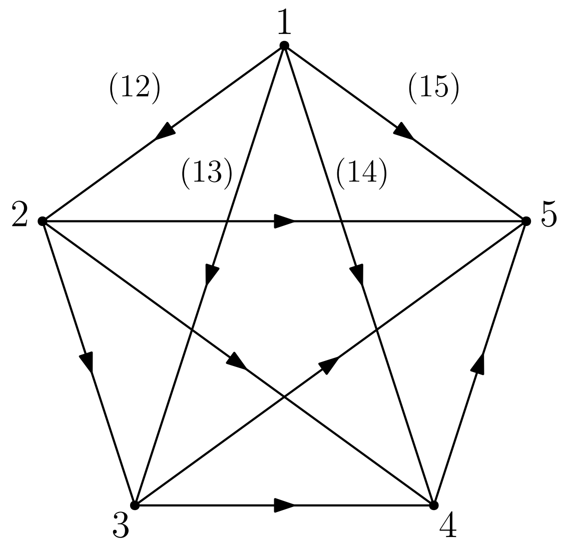
Geometry intrinsically associated to the holonomy

(*similar* to twisted geometries, but I am not talking about fluxes)

Setting the scene for local flatness

Consider the 2-complex of one 4-simplex (no geometric data, just combinatorial information)



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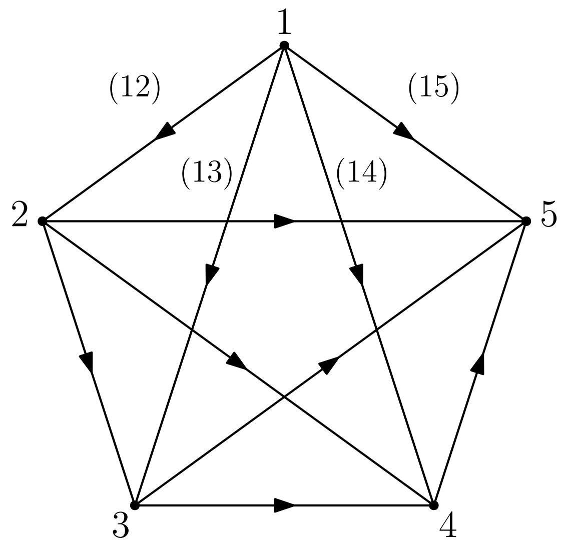

Five edges (dual to tetrahedra)

a=1, 2, 3, 4, 5

Ten faces (wedges) given by a couple of edges

(ab)=(12), (13), (14), (15), ...

Associate one $SL(2,\mathbb{C})$ holonomy to each wedge

$$g_{ab} = e^{\frac{\omega_{ab}}{2}} |z_{ba}]\langle z_{ab}| - e^{-\frac{\omega_{ab}}{2}} |z_{ba}\rangle [z_{ab}|$$

(changed convention slightly, helps with interpretation)

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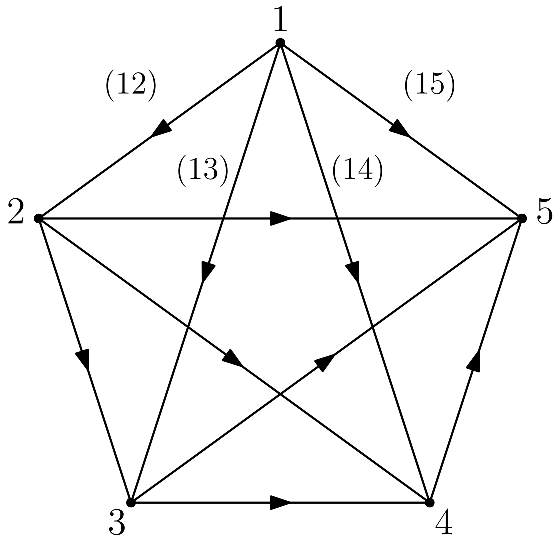

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Ten holonomies describe the parallel transport from one edge to another. They associate four framed planes to each edge and a complex number to each face.

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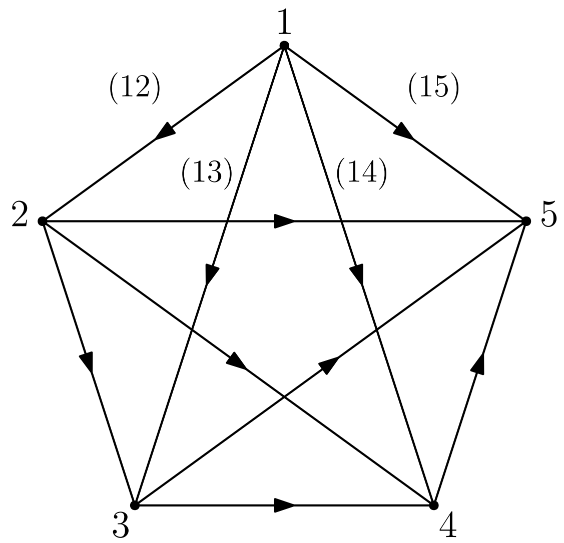

Tetrahedra?

Holonomies knows about angles.

No areas! No closure condition!

(SU(2) edge invariance. Info about fluxes. If confusing think about them as tetrahedra. They are one closure condition away)

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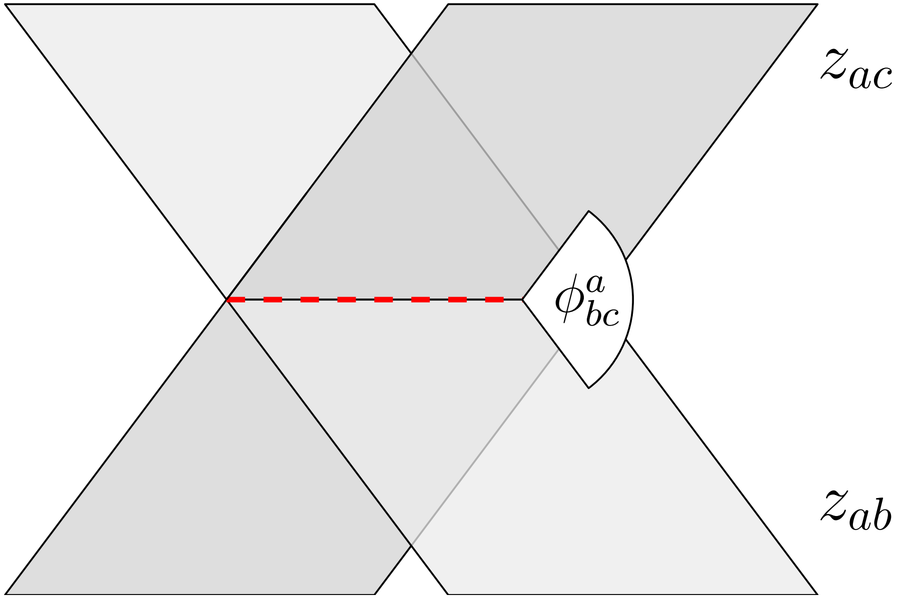
Some geometric quantities

3d dihedral angles
(angle between 2 framed planes at the same edge)

$$\cos\phi^a_{bc} = \vec{n}_{ab} \cdot \vec{n}_{ac} = 2|\langle z_{ab}|z_{ac}\rangle|^2 - 1$$



Some geometric quantities

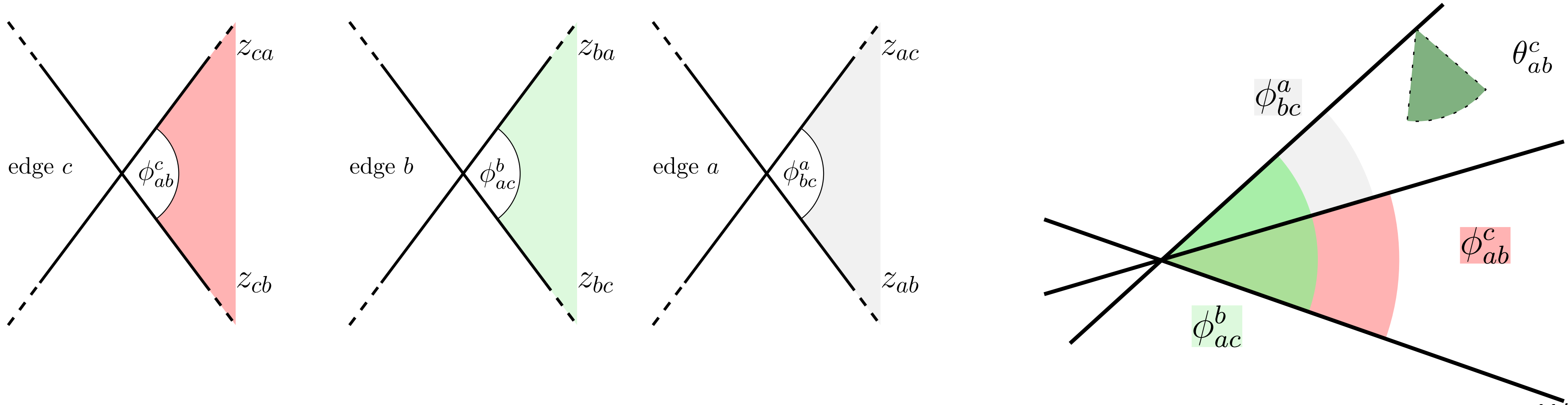
Spherical cosine law and sine law

(local embedding of 3D hyperplanes in 4D – signature?)

$$\cos\hat{\theta}^c_{ab} = \frac{-|\langle z_{ca}|z_{cb}]|^2 + |\langle z_{ab}|z_{ac}\rangle|^2|\langle z_{ba}|z_{bc}\rangle|^2 + |\langle z_{ac}|z_{ab}]|^2|\langle z_{ba}|z_{bc}]|^2}{2|\langle z_{ac}|z_{ab}\rangle\langle z_{ac}|z_{ab}]\langle z_{ba}|z_{bc}\rangle\langle z_{ba}|z_{bc}]|} = \frac{\cos\phi^c_{ab} + \cos\phi^a_{bc}\cos\phi^b_{ac}}{\sin\phi^a_{bc}\sin\phi^b_{ac}}$$

$$\sin\phi^b_{ac}\sinh\hat{\theta}^c_{ab} = \sin\phi^c_{ab}\sinh\hat{\theta}^b_{ac}$$

(schematic picture in 1 lower dimension – I cannot draw in 4D)

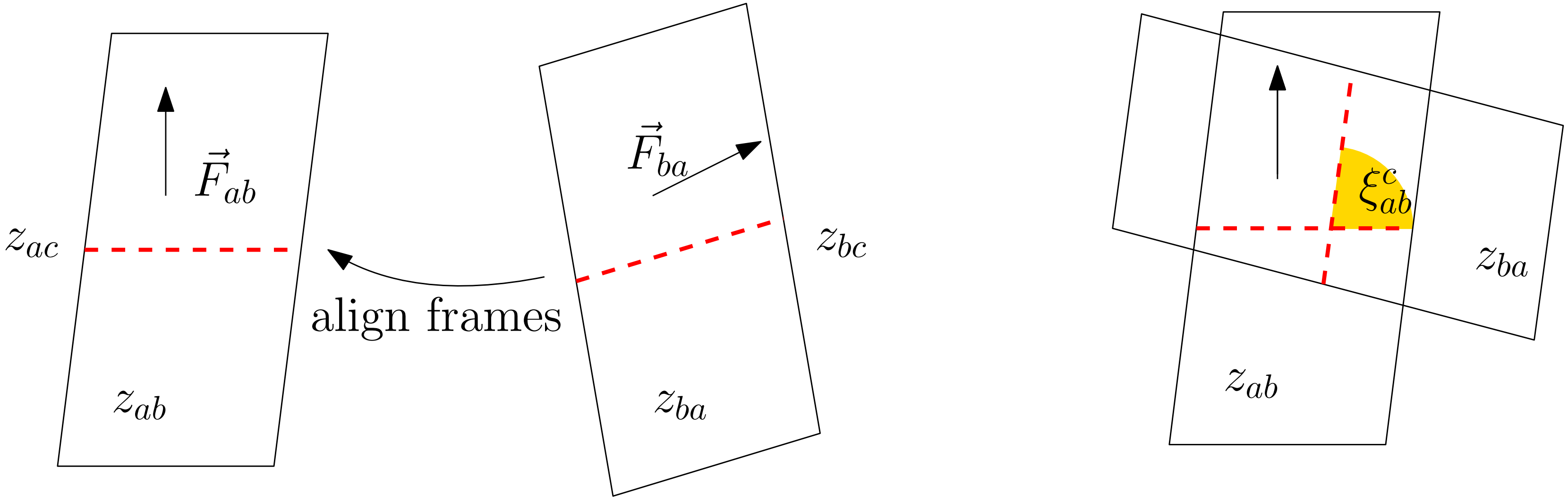


Some geometric quantities

Twist angle

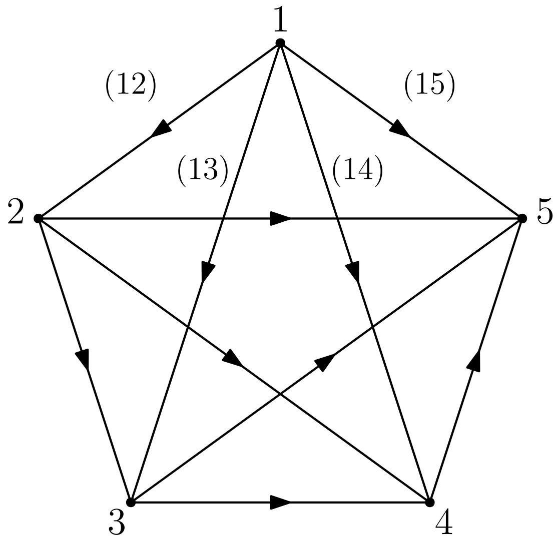
(measure twist between frames using a third as reference -
the same one defined by Bianca and Jimmy or Simone and Fabio)

$$\xi_{ab}^{c} = \arg\left(\frac{\langle z_{ac}|z_{ab}]\langle z_{ab}|z_{ac}\rangle}{\langle z_{bc}|z_{ba}\rangle[z_{ba}|z_{bc}\rangle}\right)$$



Holonomies and local flatness

Flat building blocks!



We require each 4-simplex to be flat! The 2-complex is locally flat.

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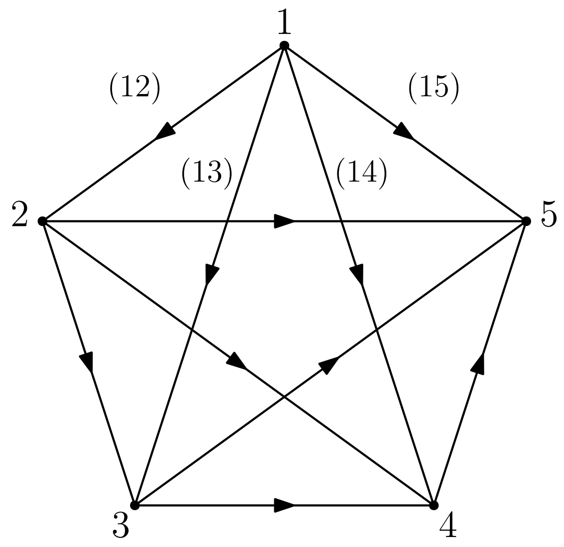

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In terms of holonomies the parallel transport on every closed cycle in the 4-simplex is trivial

$$g_{ca}g_{bc}g_{ab} = \mathbb{1}$$

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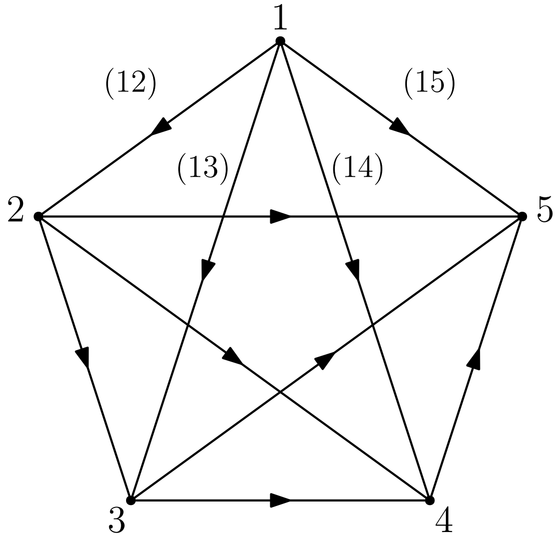

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Constraints on holonomies = constraints on the geometries (framed planes, spinors and complex angles)

How to solve? Smart projection on components

$$[z_{ac}|g_{ab}^{-1}|z_{bc}\rangle = [z_{ac}|g_{ca} g_{bc}|z_{bc}\rangle$$
$$\langle z_{ac}|g_{ab}^{-1}|z_{bc}] = \langle z_{ac}|g_{ca} g_{bc}|z_{bc}]$$

Plus other two. 4 complex scalar equations.
Combine to find...

Holonomies and local flatness

Local flatness

$$g_{ca} g_{bc} g_{ab} = \mathbb{1}$$

=

Complex angle determined by spinors

$$\cosh(\omega_{ab} + i\xi^c_{ab}) = \cos \hat{\theta}^c_{ab}$$

$$\sin \phi^b_{ac} \sinh(\omega_{ab} + i\xi^c_{ab}) = \sin \phi^c_{ab} \sinh(\omega_{ca} + i\xi^b_{ac})$$

for every cycle = constrains also the spinors!

Solutions?

Studied by Me and Simone 2 years ago

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Solutions?

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Lorentzian sector $\quad |\cos\hat{\theta}^c_{ab}| > 1$

The other sector is the topological one (vector and euclidean) with $\mathrm{SU}(2)$ holonomies

Lorentzian geometries (need edge independence = angle matching = “shape” matching)

$$\omega_{ab} = \epsilon\theta_{ab} + i\epsilon\chi_{ab}\pi - i\xi_{ab}$$

orientation, dihedral angle, local causal structure, twist angles

Holonomies and local flatness

Local flatness

$$g_{ca}g_{bc}g_{ab} = \mathbb{1}$$

$=$

Complex angle determined by spinors

+

angle matching conditions (strongly)

(restriction to the Lorentzian sector)

$$\omega_{ab} = \epsilon\theta_{ab} + i\epsilon\chi_{ab}\pi - i\xi_{ab}$$

+

closure conditions

(areas up to a scale)

$$\sum_{b\neq a} j_{ab}\vec{n}_{ab} = 0$$

Information about fluxes, related to edge SU(2) invariance

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Information about fluxes, related to edge SU(2) invariance

In ALL locally flat Lorentzian spin foam models

(with edge SU(2) invariance)

shape matched Lorentzian 4-simplices emerge!

General! No amplitude, embedding map, semiclassical regime or critical point eqs.

Local flatness in the EPRL model

(all the spin foam models are locally flat)

$$A_v = \int \prod_a \mathrm{d}g_a \delta(g_1) \prod_{ab} D^{(\gamma j_{ab}, j_{ab})}_{j_{ab}\zeta_{ba} j_{ab}\zeta_{ab}} (g_b^{-1} g_a)$$

(edge holonomies)

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(edge holonomies)

The EPRL model is locally flat (imposed strongly)

$$A_v = \int \left(\prod_{ab} \mathrm{d}g_{ab} D^{(\gamma j_{ab}, j_{ab})}_{j_{ab}\zeta_{ba} j_{ab}\zeta_{ab}} (g_{ab}) \right) \mathcal{C}_{LF}(g_{ab}, \cdots, g_{cd})$$

(wedge holonomies)

$+$

$$\begin{aligned}\mathcal{C}_{LF}(g_{ab}, \cdots, g_{cd}) =& \delta\left(g_{13}^{-1} g_{23} g_{12}\right) \delta\left(g_{14}^{-1} g_{24} g_{12}\right) \delta\left(g_{15}^{-1} g_{25} g_{12}\right) \cdot \\ & \delta\left(g_{14}^{-1} g_{34} g_{13}\right) \delta\left(g_{15}^{-1} g_{35} g_{13}\right) \delta\left(g_{15}^{-1} g_{45} g_{14}\right) \ .\end{aligned}$$

Curiosity: the necessity of regularize the amplitude correspond to the necessity of consider fundamental cycles!
Different regularization choice = different fundamental cycles choice

What is the role of the critical point equations?

$$A_v = \int \left(\prod_{ab} \mathrm{d}g_{ab} D^{(\gamma j_{ab}, j_{ab})}_{j_{ab}\boxed{\zeta_{ba}} j_{ab}\boxed{\zeta_{ab}}}(g_{ab}) \right) \mathcal{C}_{LF}(g_{ab}, \cdots, g_{cd})$$

Coherent boundary data allow evaluation of the amplitude's integrals at the saddle point. ⟶ critical point equations

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Coherent boundary data allow evaluation of the amplitude’s integrals at the saddle point. → critical point equations

Closure conditions of the boundary data (consequence of the SU(2) invariance of the amplitude) → Framed planes at the same edge close forming a framed tetrahedron (areas given by the spins)

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Closure conditions of the boundary data (consequence of the SU(2) invariance of the amplitude) → Framed planes at the same edge close forming a framed tetrahedron (areas given by the spins)

Alignment equations → The spinors (framed planes) parametrizing the holonomies and the boundary ones are the same

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Saddle point → **Closure conditions** (boundary described as framed tetrahedra) + **Alignment equations** (holonomies spinors coincide with boundary ones)

Action at the critical point

$$i\lambda \sum_{ab} j_{ab} \left(\gamma \mathrm{Re}\, \omega_{ab} + \mathrm{Im}\, \omega_{ab} \right)$$

The connection with the Regge-Action happens only on-shell of the local flatness conditions

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focus to the Lorentzian sector

$$\omega_{ab} = \boxed{\epsilon\theta_{ab}} + \boxed{i\epsilon\chi_{ab}\pi} - \boxed{i\xi_{ab}}$$

4D dihedral angle between the framed tetrahedra (ab)

Local causal structure

Twist between the framed tetrahedra (ab)

SKIP

Other stuffs

Many vertices?

Analysis on vertex amplitudes independently
(local flatness + closure + alignment)

$+$

Extra alignment equations
(framed tetrahedra shared by different vertices coincide!)

Summing over the spins

Constraining face holonomies?

Singular support of the face distribution

(mystic result by Hellmann and Kaminski)

Naive flatness problem arises when you combine
Local flatness + singular support + alignment

Topological BF: $\delta(g_f) \longrightarrow g_f = \mathbb{1}$

EPRL: $f_{EPRL}(g_f) \longrightarrow g_f = e^{\frac{\omega_f}{2}} |\zeta\rangle\langle\zeta| + e^{-\frac{\omega_f}{2}} |\zeta][\zeta| \text{ with } \gamma \text{Re}\omega_f + \text{Im}\omega_f = 0 \text{ mod } 4\pi$

Conclusions

Local flatness is responsible of the local emergence of Regge geometry in spin foam models

Integrating over locally flat holonomies with $\mathrm{SU}(2)$ edge invariance restricted on the Lorentzian sector = Summing over Lorentzian 4-simplices

(effective spin foam models and area-angle Regge calculus)

Quantum simplicity constraints (alignment + action)

Secondary simplicity constraints? (shape matching, imposed strongly)

Separation of ingredients is key to innovate (maybe new model? Simpler to do calculations! Top down construction! I have no concrete proposal.)

End